Transasymptotics, dynamical systems and far from equilibrium hydrodynamics

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BEST COLLABORATION

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Hydrodynamics: one theory to rule them all

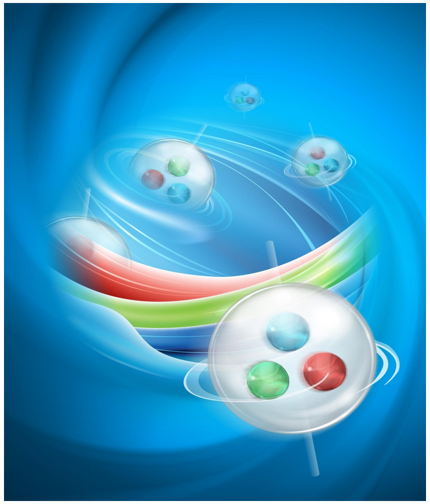
Water

Ketchup

Olive oil

Coffee

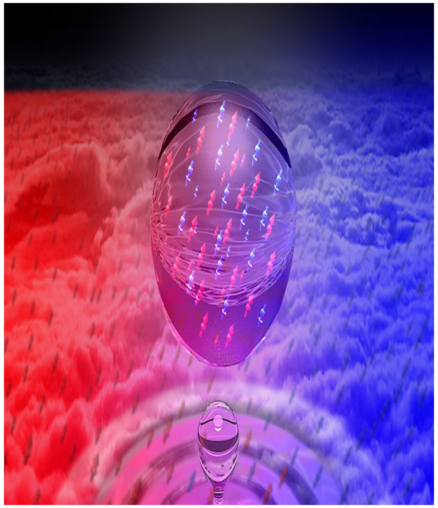
Quark-Gluon Plasma



$T \sim 10^{12}\,K$

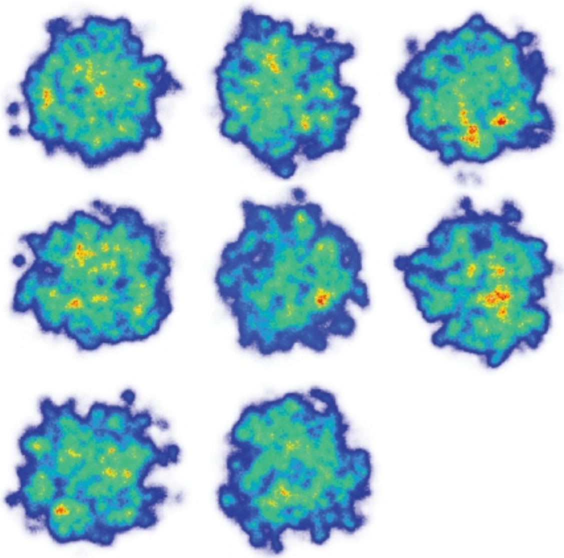
New discoveries: Nearly Perfect Fluids

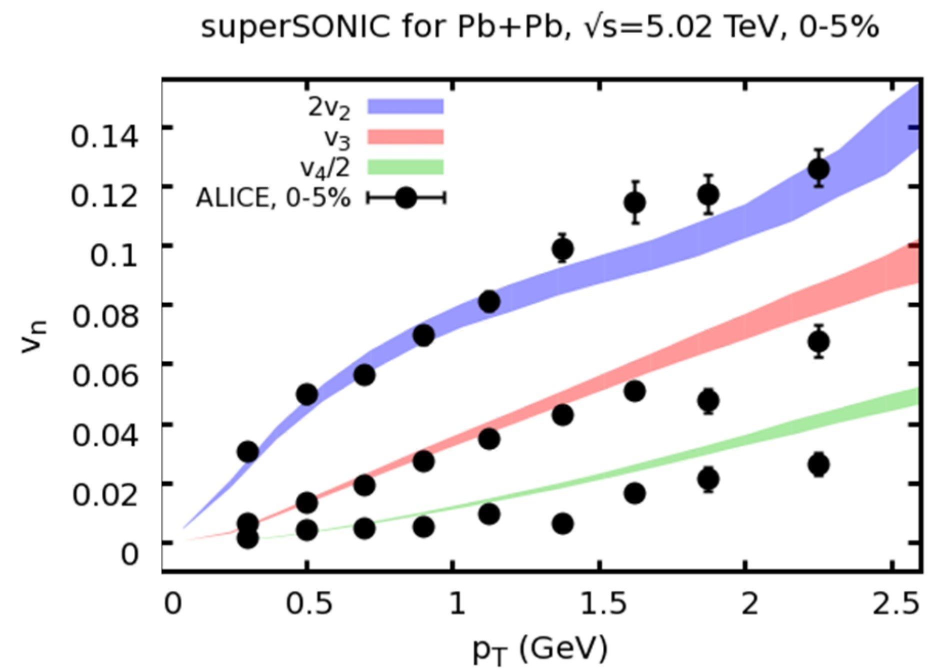
Ultracold atoms



$T \sim 10^{-7}\,K$

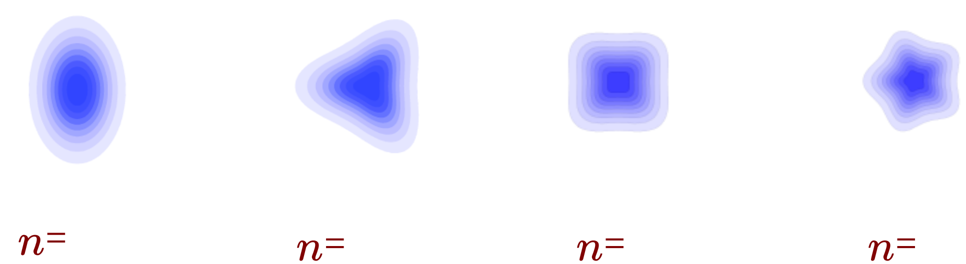
Fluidity in Heavy Ions





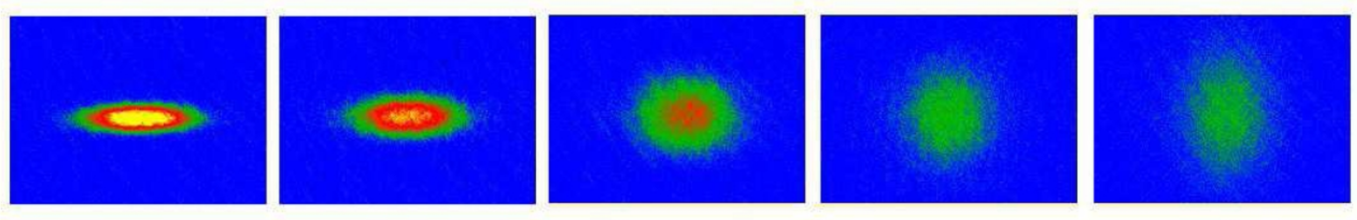
Weller & Romatschke (2017)

$$\frac{dN}{d\phi} = \frac{N}{2\pi}\left(1 + \sum_n (2v_n \cos(n\phi))\right)$$

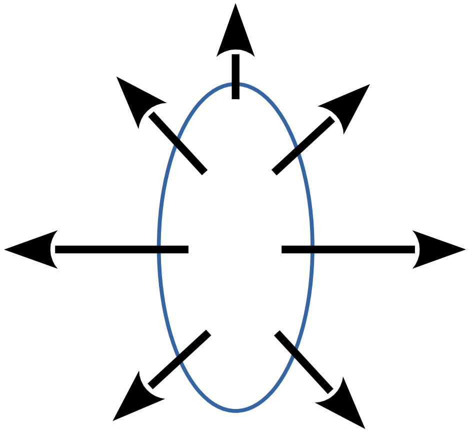


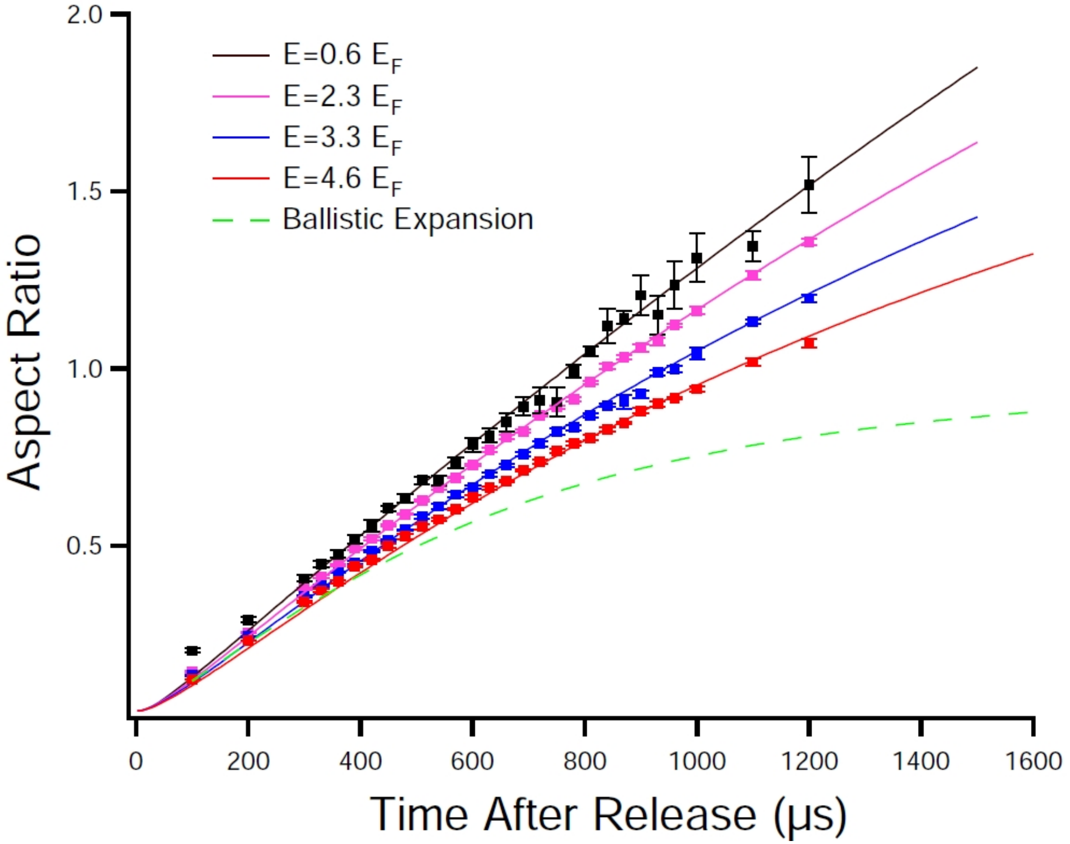
v_n provides information of the initial spatial geometry of the collision

Fluidity in Cold Atoms



Aspect ratio measures pressures anisotropies





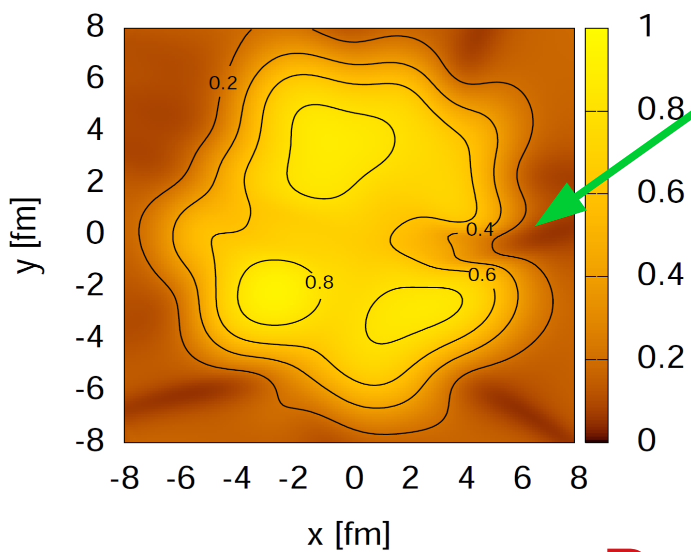
Cao et. al (2010)

Size of the hydrodynamical gradients

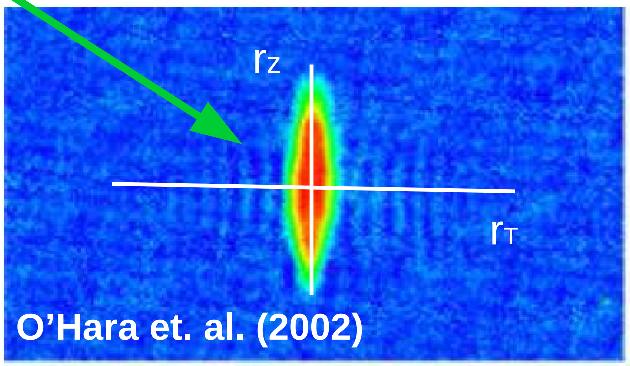
Heavy Ion Collisions

Martinez et. al. (2012)

P_L/P_T at τ = 2.50 fm/c



Cold Atoms



Pressure anisotropies are not small

Paradox:

Hydrodynamics provides a good description despite large pressure anisotropies.

Introductory textbook:

Hydrodynamics is valid as far as the system is near equilibrium

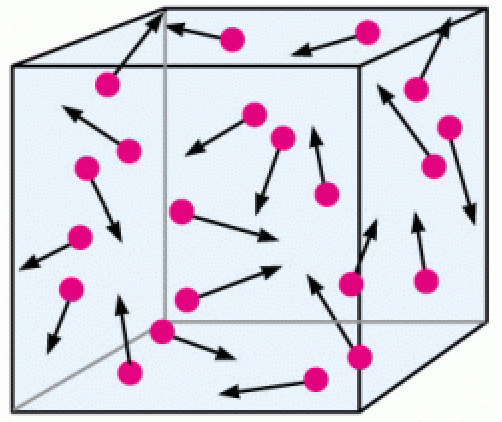
How does hydro emerges from a non-equilibrium initial state?

Far-from-equilibrium

↓ **?**

Hydrodynamics

A bit of kinetic theory



Boltzmann equation

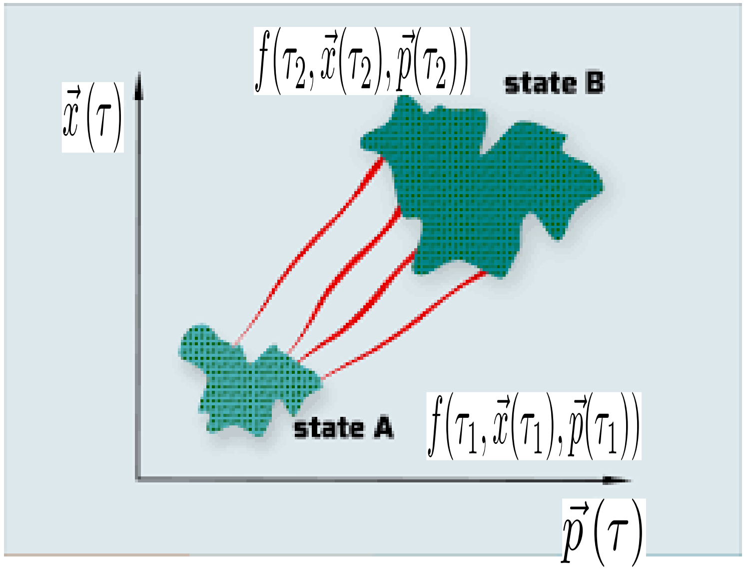
Microscopic dynamics is encoded in the distribution function f(t,**x**,**p**)

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x^i} + F^i \frac{\partial f}{\partial p^i} = -\mathcal{C}[f]$$

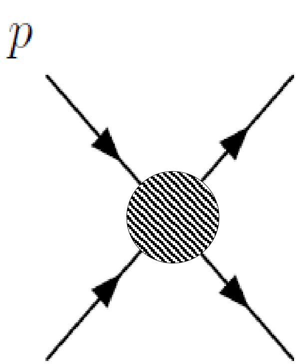
Diffusion

External Force

Particle imbalance

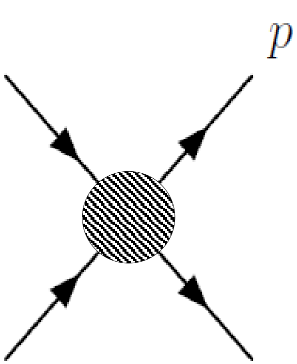


$$\mathcal{C}[f] =$$

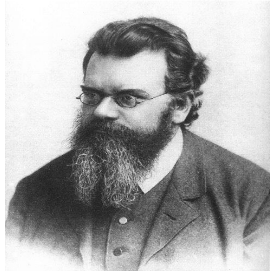


Gain

-



Lose



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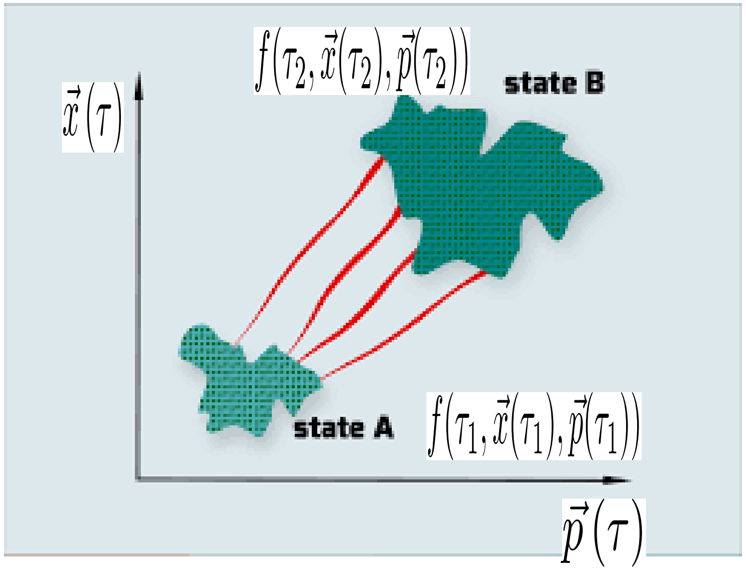
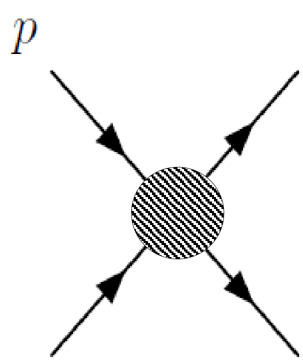
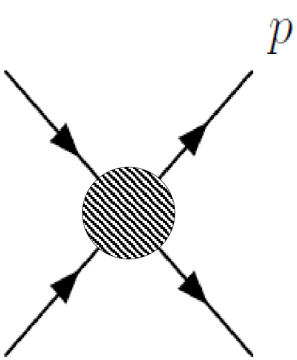


$$\mathcal{C}[f] =$$



Gain

-



Lose

Observables

Macroscopic quantities are simply averages , e.g.,

$$T^{\mu\nu} = \int_{\mathbf{p}} p^\mu p^\nu f(x^\mu, \mathbf{p})$$

Near to equilibrium

$$T^{\mu\nu} = \sum_{k=0}^{\infty} (Kn)^k T_k^{\mu\nu} \qquad Kn \equiv \frac{l}{L}$$

Energy-momentum tensor of a viscous fluid

$$T_0^{\mu\nu} = (\epsilon + p(\epsilon))\, u^\mu u^\nu + p(\epsilon) g^{\mu\nu}$$ → Ideal fluid $\mathcal{O}(\mathrm{Kn}^0)$

$$T_1^{\mu\nu} = -\eta\, \sigma^{\mu\nu}$$ → $\mathcal{O}(\mathrm{Kn})$: Navier-Stokes

$$T_2^{\mu\nu}$$ → $\mathcal{O}(\mathrm{Kn}^2)$: IS, etc

Asymptotics in the Boltzmann equation

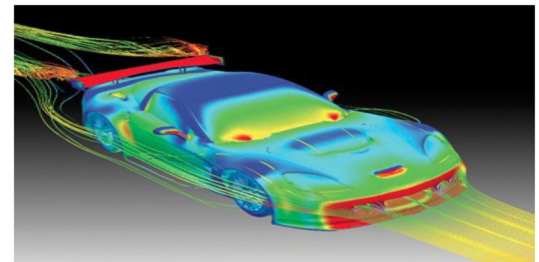
Usually the distribution function is expanded as series in Kn, i.e.,

$$f(x^\mu, p) = \sum_{k=0}^{\infty} (Kn)^k \, f_k(x^\mu, p)$$

$$Kn \equiv \frac{l}{L}$$

Microscopic scale (Mean free path) $\quad l \sim \lambda_{mfp}$

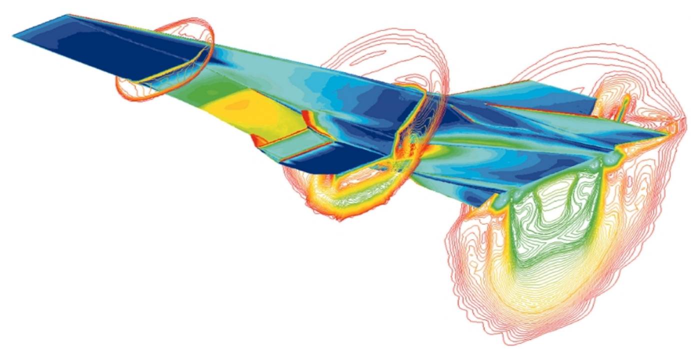
Macroscopic scale (spatial gradients) $\quad \frac{1}{L} \sim \partial_i v^i$



$L \sim 1\,m \qquad \ell \sim 10^{-7}\,m$

Expansion fails if

$$Kn \sim \frac{l}{L} \sim 1$$



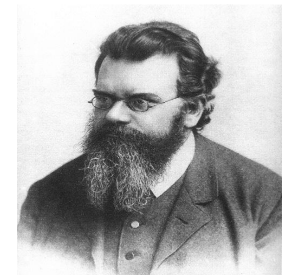
High-altitude flights

Dryden Flight Research Center ED97 43968-01
HYPER-X AT MACH 7: This computational fluid dynamic (CFD) image is of the Hyper-X vehicle at the Mach 7 test condition with the engine operating.

Kinetic theory: Boltzmann equation

Grad’s moments method

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x^i} + F^i \frac{\partial f}{\partial p^i} = -\mathcal{C}[f]$$



$$f(x^\mu, \mathbf{p}) = f_0 \left(1 + \sum_{l=0}^{\infty} \sum_{n=0}^{N_l} \mathcal{H}^l_{\mathbf{p},n} \, \rho_n^{\mu_1 \mu_2 \cdots \mu_l} \, p_{\langle \mu_1} \cdots p_{\mu_l \rangle} \right)$$

Background distribution

Polynomials of energy

Irreducible moments

Irreducible tensors

Grad (1949), Israel-Stewart (1976), DNMR (2010)

Kinetic theory: Boltzmann equation

Grad's moments method

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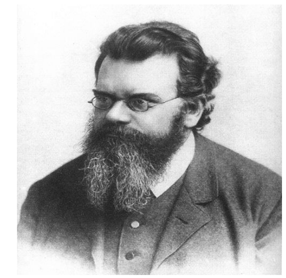


$$f(x^\mu, \mathbf{p}) = f_0 \left(1 + \sum_{l=0}^{\infty} \sum_{n=0}^{N_l} \mathcal{H}^l_{\mathbf{p},n} \, \rho_n^{\mu_1 \mu_2 \cdots \mu_l} p_{\langle \mu_1} \cdots p_{\mu_l \rangle} \right)$$

Relaxation to the asymptotic state of the distribution function is determined by analyzing the non-linear evolution equation of the moments

$$\frac{d\rho_r^{\mu_1 \mu_2 \cdots \mu_l}}{dt} \sim \frac{d}{dt} \left[\int_{\mathbf{p}} E^r_{\mathbf{p}} \, p^{\langle \mu_1} \cdots p^{\mu_l \rangle} \delta f \right]$$

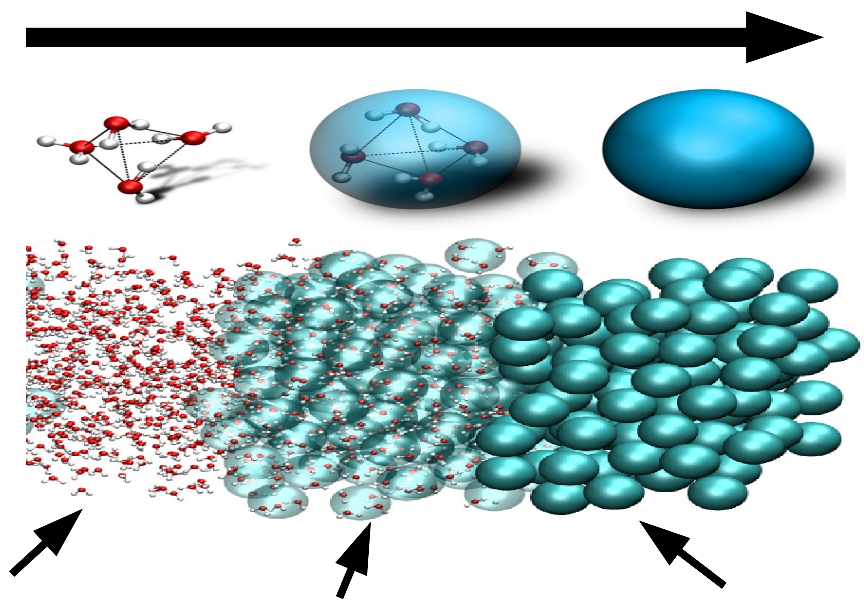
Grad (1949), Israel-Stewart (1976), DNMR (2010)

Hydro as an coarse-grained approach

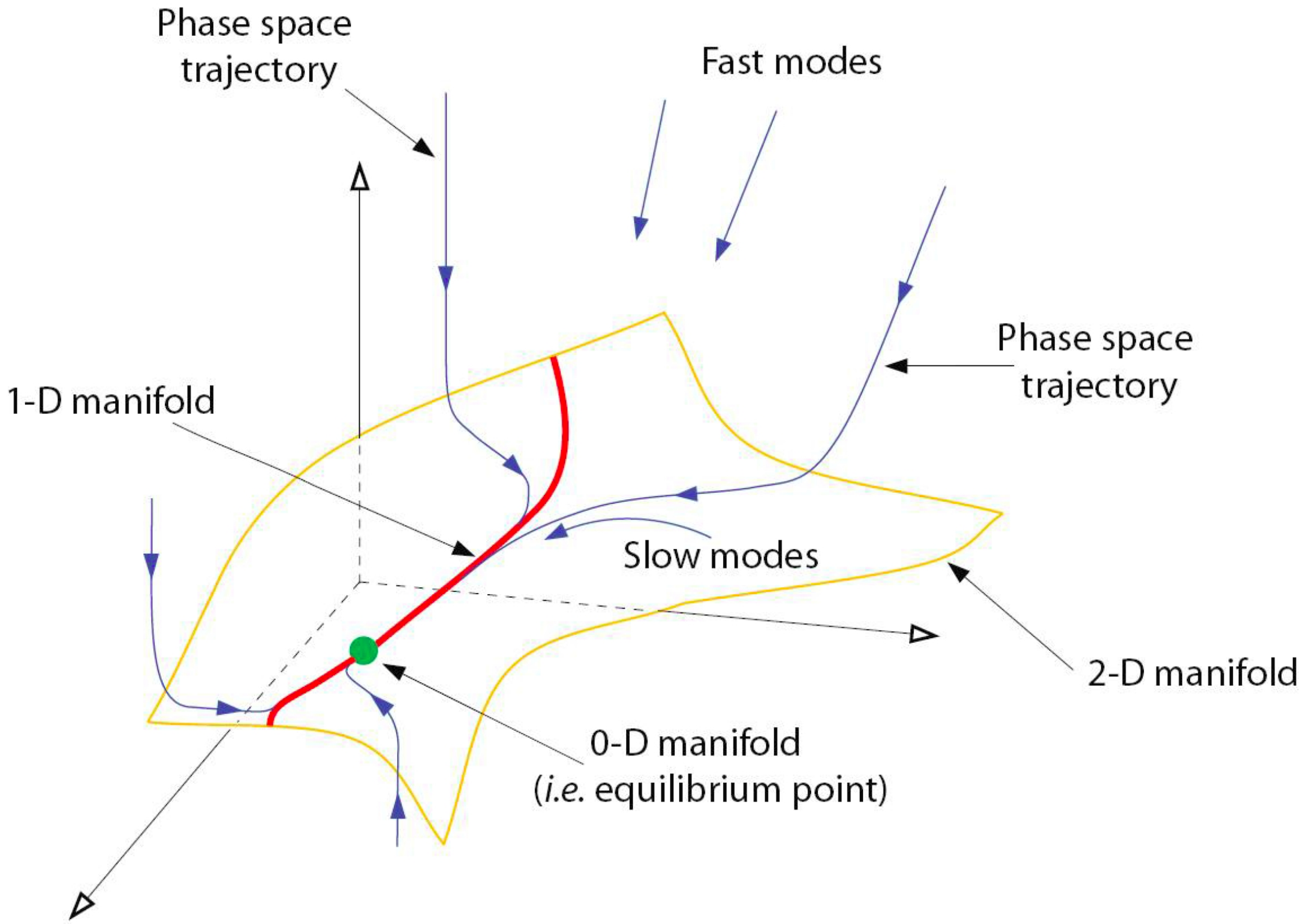
How many moments do we need?

$$f(x^\mu, \mathbf{p}) = f_0 \left(1 + \sum_{l=0}^{\infty} \sum_{n=0}^{N_l} \mathcal{H}^{l}_{\mathbf{p},n} \, \rho_n^{\mu_1 \mu_2 \cdots \mu_l} p_{\langle \mu_1} \cdots p_{\mu_l \rangle} \right)$$

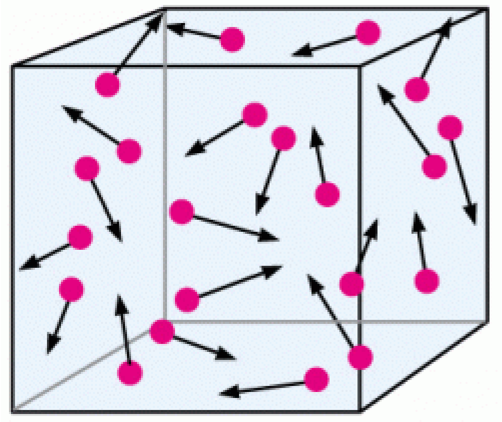
- **Coarse-grained procedure reduces # of degrees of freedom**
- **The slowest degrees of freedom determine hydrodynamics**
- **However, kinetic theory is highly non-linear…..**



Slow invariant manifold picture



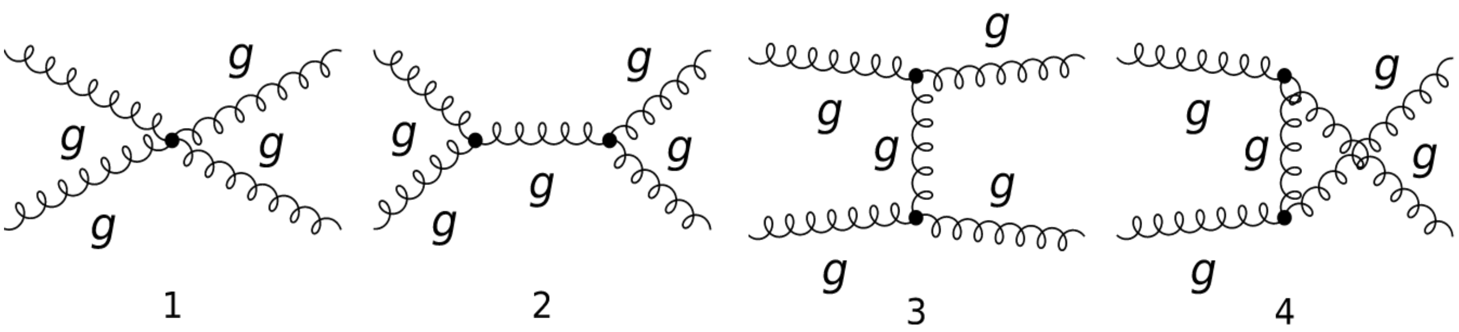
Fokker-Planck equation



Diffusive approximation

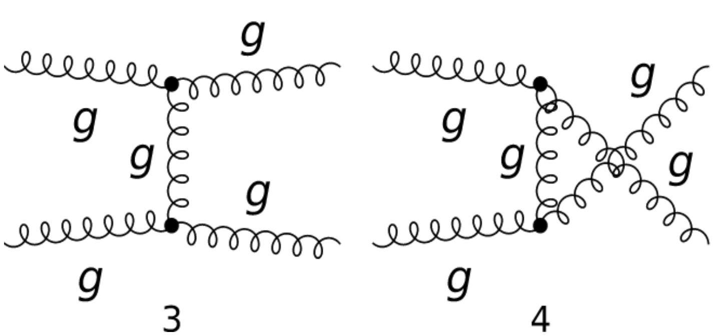
$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x^i} = -\mathcal{C}[f]$$

$$\mathcal{C}[f] =$$



Within the small angle approximation

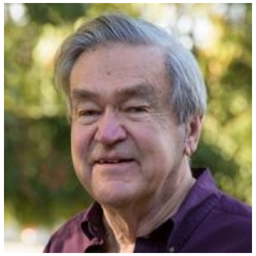
$$\mathcal{C}_{diff.}[f] \approx$$



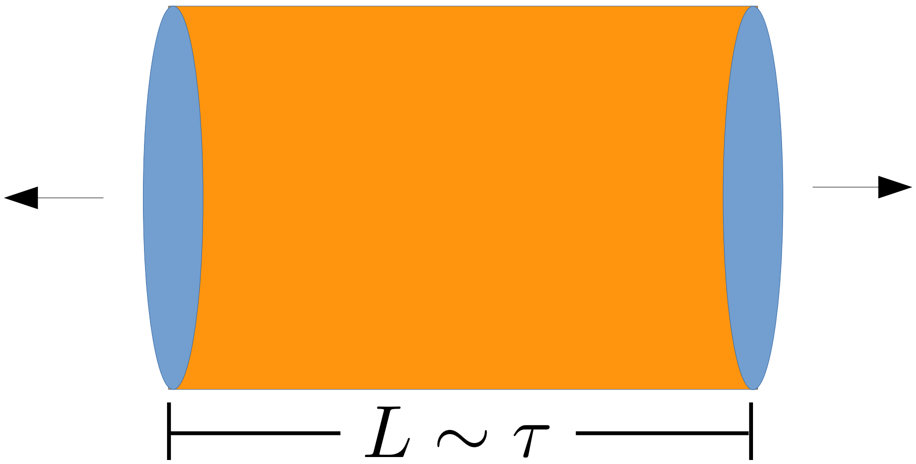
Landau & Lifschitz, Physical kinetics
For gluons: A. Mueller (1999)

Bjorken flow expansion

Toy model: expanding system which is longitudinally boost invariant, **Bjorken flow** (Bjorken 1983)



Near equilibrium one can calculate the coefficients in the perturbative expansion



$$T^{\mu\nu} = \sum_{k=0}^{\infty} T_k^{\mu\nu} \, [\mathrm{Kn}]^k$$

$$\mathrm{Kn} \sim \frac{1}{T\,\tau} \equiv \frac{1}{w}$$

Instead of solving the Boltzmann equation we study the dynamical equations of the moments

Expansion in moments for Bjorken flow

$$\partial_\tau f(\tau, p_T, p_z) = \mathcal{C}_{diff.}[f]$$

By expanding the distribution function in orthogonal polynomials

$$f(x, \mathbf{p}) = f_{eq.}\left(E_\mathbf{p}/T(\tau)\right) \sum_{l=0}^{\infty} c_l(\tau)\, \mathcal{P}_{2l}(\cos\theta_\mathbf{p})$$

Physical observables:

$$T^{\mu\nu} = \int_\mathbf{p} p^\mu p^\nu f(x^\mu, \mathbf{p}) \equiv \text{ diag.}\left(\epsilon, P_T, P_T, P_L\right)$$

$$\epsilon \sim T^4 \qquad P_T = \epsilon\left(\frac{1}{3} - \frac{c_1}{15}\right) \qquad P_L = \epsilon\left(\frac{1}{3} + \frac{2}{15}c_1\right)$$

Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Expansion in moments for Bjorken flow

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The problem of solving the FP Eqn is mapped into solving a nonlinear ODEs for the Legendre moments

$$\mathrm{Kn} \sim \frac{1}{T\,\tau} \equiv \frac{1}{w}$$

$$\frac{d\mathbf{c}}{dw} = \mathbf{F}(\mathbf{c}, w),$$

$$\mathbf{F}(\mathbf{c}, w) = -\frac{1}{1 - \frac{1}{20} c_1(w)} \left[\frac{1}{w} \left\{ \mathfrak{X}(\mathbf{c})\mathbf{c}(w) + \mathbf{\Gamma} \right\} + \left\{ \Lambda + \mathfrak{Y}(\mathbf{c}) + \mathfrak{Z}(\mathbf{c}) \right\} \mathbf{c}(w) \right].$$

Martinez et. al. 2011.08235

Non-autonomous Dynamical systems

$$\frac{d\mathbf{c}}{dw} = \mathbf{F}(\mathbf{c}, w)$$

- The evolution parameter w appears explicitly in the RHS. This is a non-autonomous dynamical system.
- When w does not appear explicitly the system is an autonomous one
- For autonomous systems the fixed points are simply dc/dw =0.
- For non-autonomous dynamical systems the invariance under translations in the w parameter is broken
- For non-autonomous dynamical systems one requires to consider limits in the past and in the future.
- These limits are not commutative.

Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Non-autonomous Dynamical systems

$$\frac{d\mathbf{c}}{dw} = \mathbf{F}(\mathbf{c}, w)$$

- Any solution, aka flow, depends on its initial value, initial and final values of w

$$\mathbf{c} \equiv \mathbf{c}(\mathbf{c}_0, w, w_0)$$

- Since future and past are not the same one requires to consider the following limits

$$\lim_{w \to \infty, w_0 fixed} \mathbf{c}(\mathbf{c}_0, w, w_0)$$

Forward Attractor

$$\lim_{w_0 \to 0, w fixed} \mathbf{c}(\mathbf{c}_0, w, w_0)$$

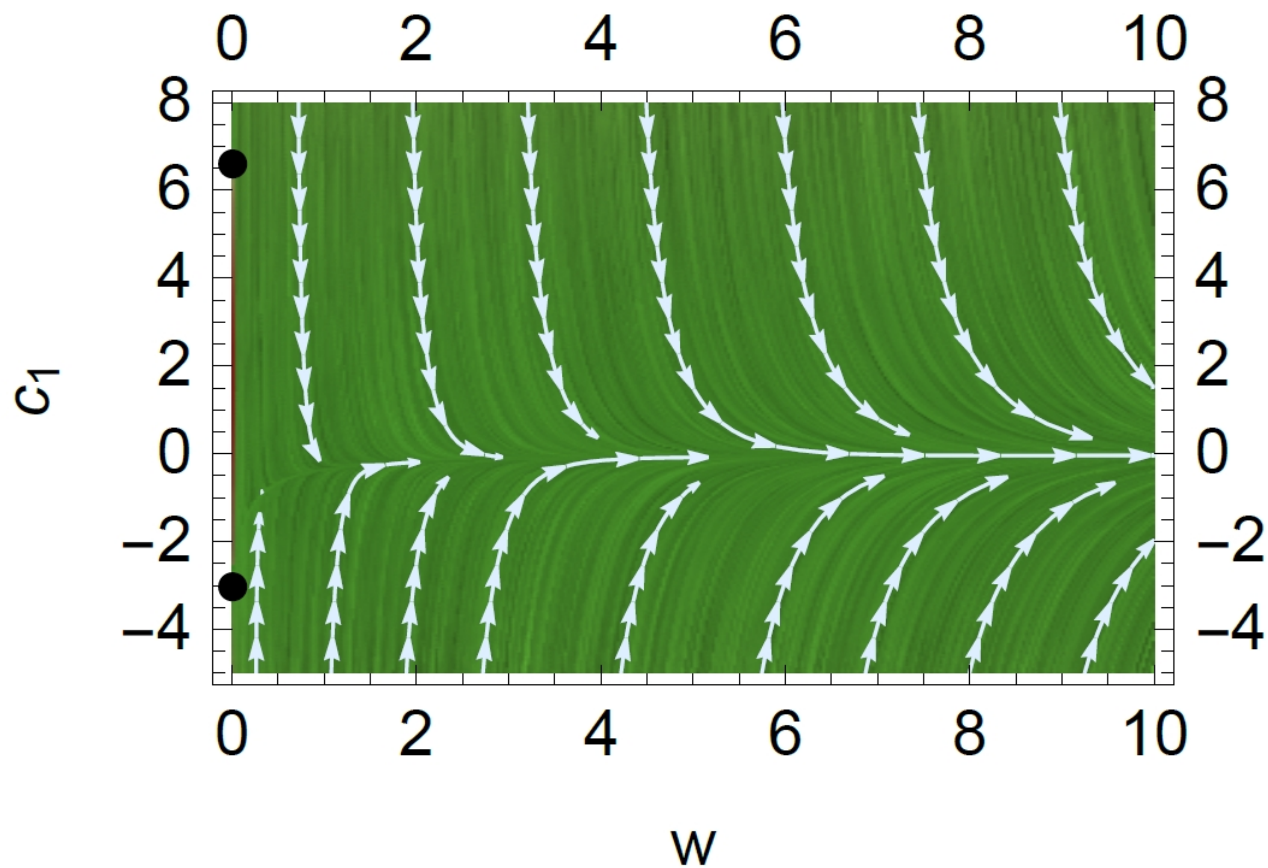
Pullback Attractor

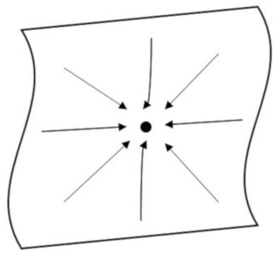
Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Basic example: flow lines in phase space

$$\frac{d\mathbf{c}}{dw} = \mathbf{F}(\mathbf{c}, w)$$

$$\lim_{w\to\infty, w_0 fixed} c_1(c_{1,0}, w, w_0) = 0$$





Sink

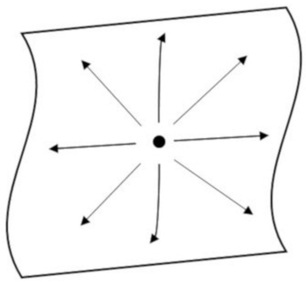
Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

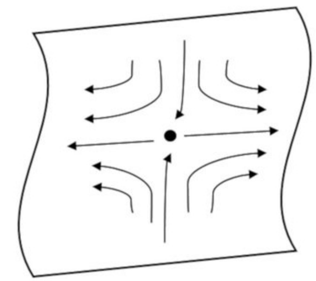
Basic example: flow lines in phase space

$$\frac{d\mathbf{c}}{dw} = \mathbf{F}(\mathbf{c}, w)$$

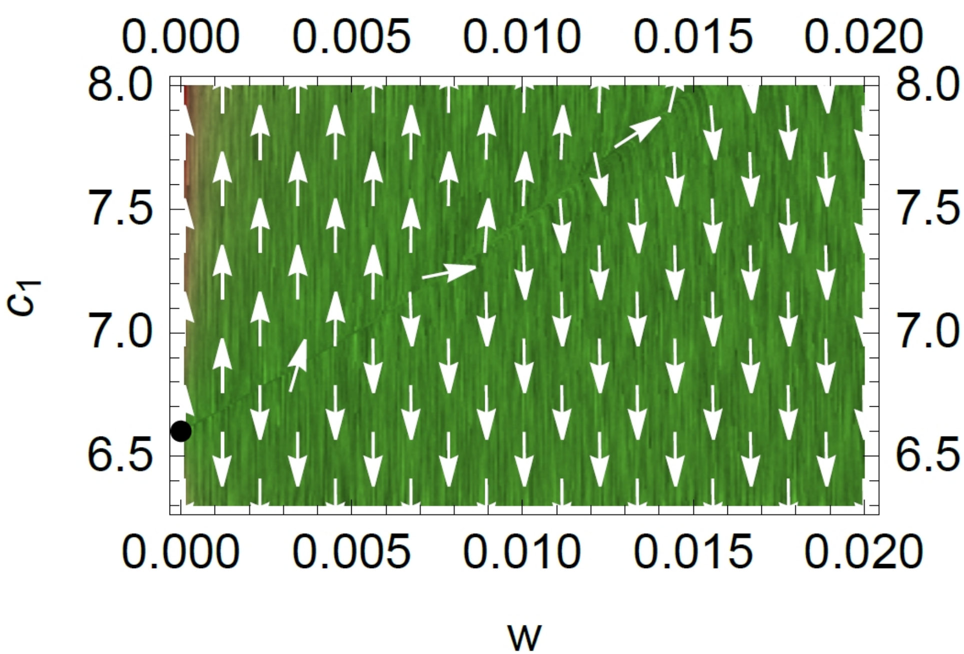
$$\lim_{w_0 \to 0, w\, fixed} \mathbf{c}(\mathbf{c}_0, w, w_0)$$

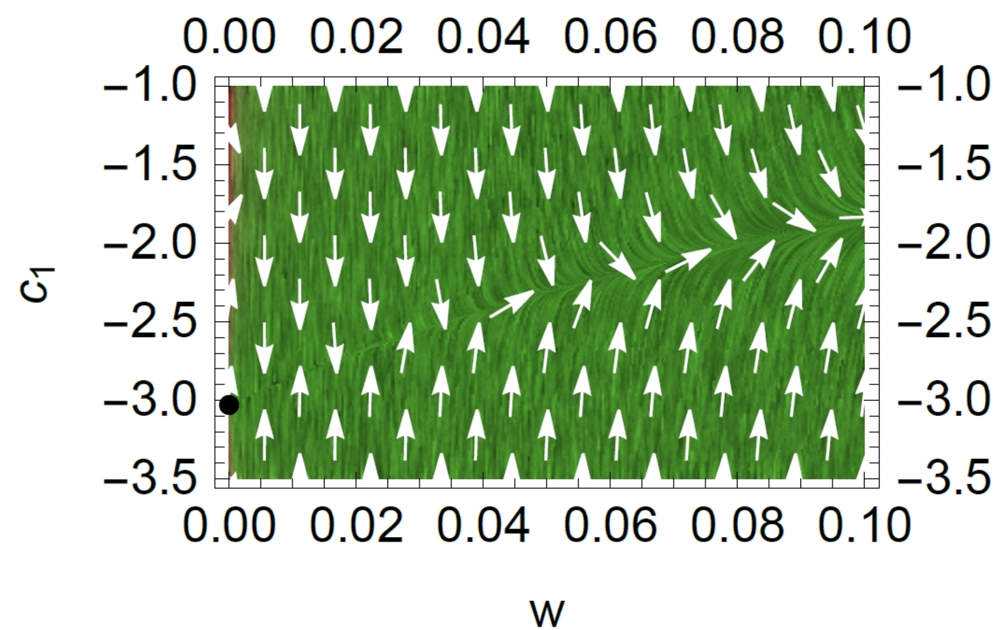
Source





Saddle





Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

UV and IR regimes

$$\frac{d\mathbf{c}}{dw} = \mathbf{F}(\mathbf{c}, w)$$

IR: w >> 1

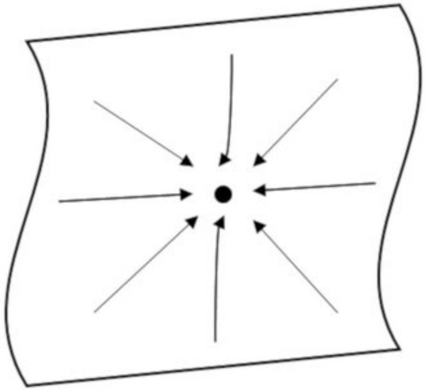
- **Near equilibrium Linear response theory**

UV: w << 1

- **Extremely far from equilibrium**

Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

IR regime



IR regime: L=1 case

$$\frac{dc_1}{dw} = F_1(w, c_1)$$

$$c_1 = \sum_{k=0}^{\infty} u_{1,k}^{(0)} \, w^{-k}$$

From linear response theory

$$c_1 = -\frac{40}{3}\frac{1}{w}\frac{\eta}{s} - \frac{80}{9}\frac{1}{w^2}\frac{T(\eta\tau_\pi - \lambda_1)}{s} \ldots$$

Transport coefficients

Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

IR regime: L=1 case

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From linear response theory

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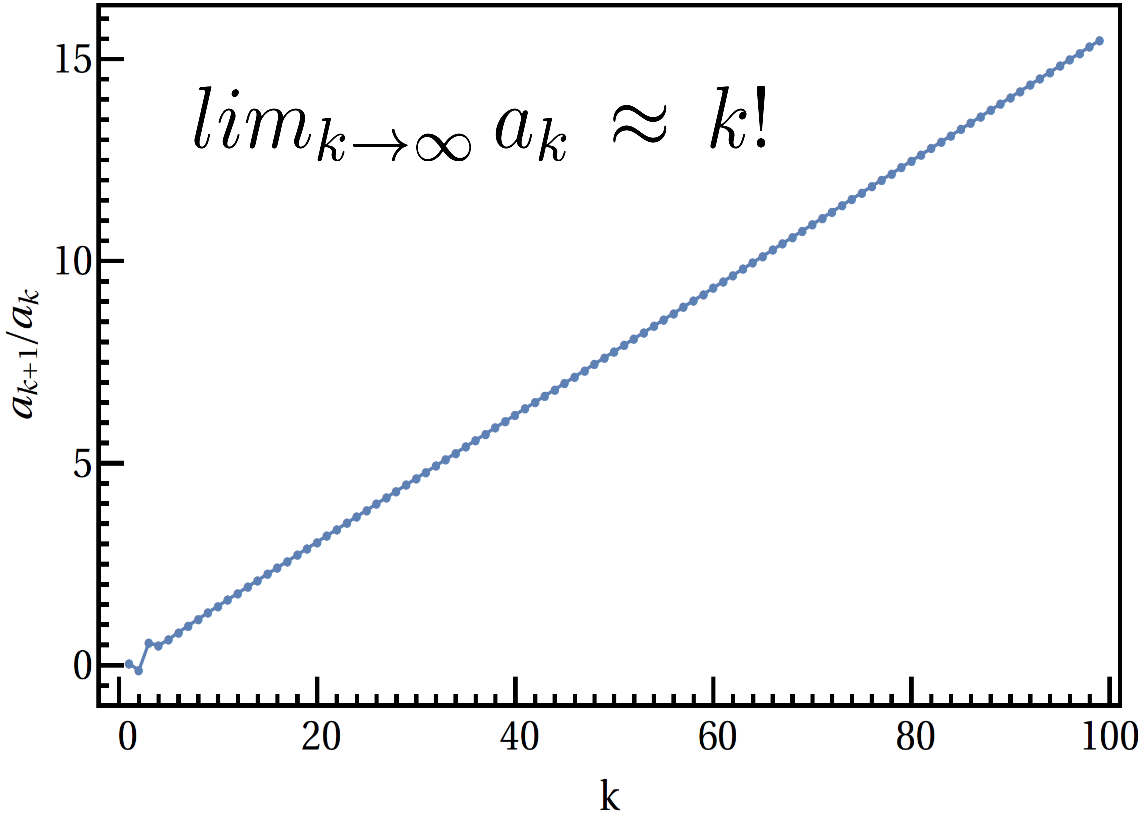

Perturbative asymptotic expansion is divergent!!!!

Borel resummation is one way to sort out this type of situations.

Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

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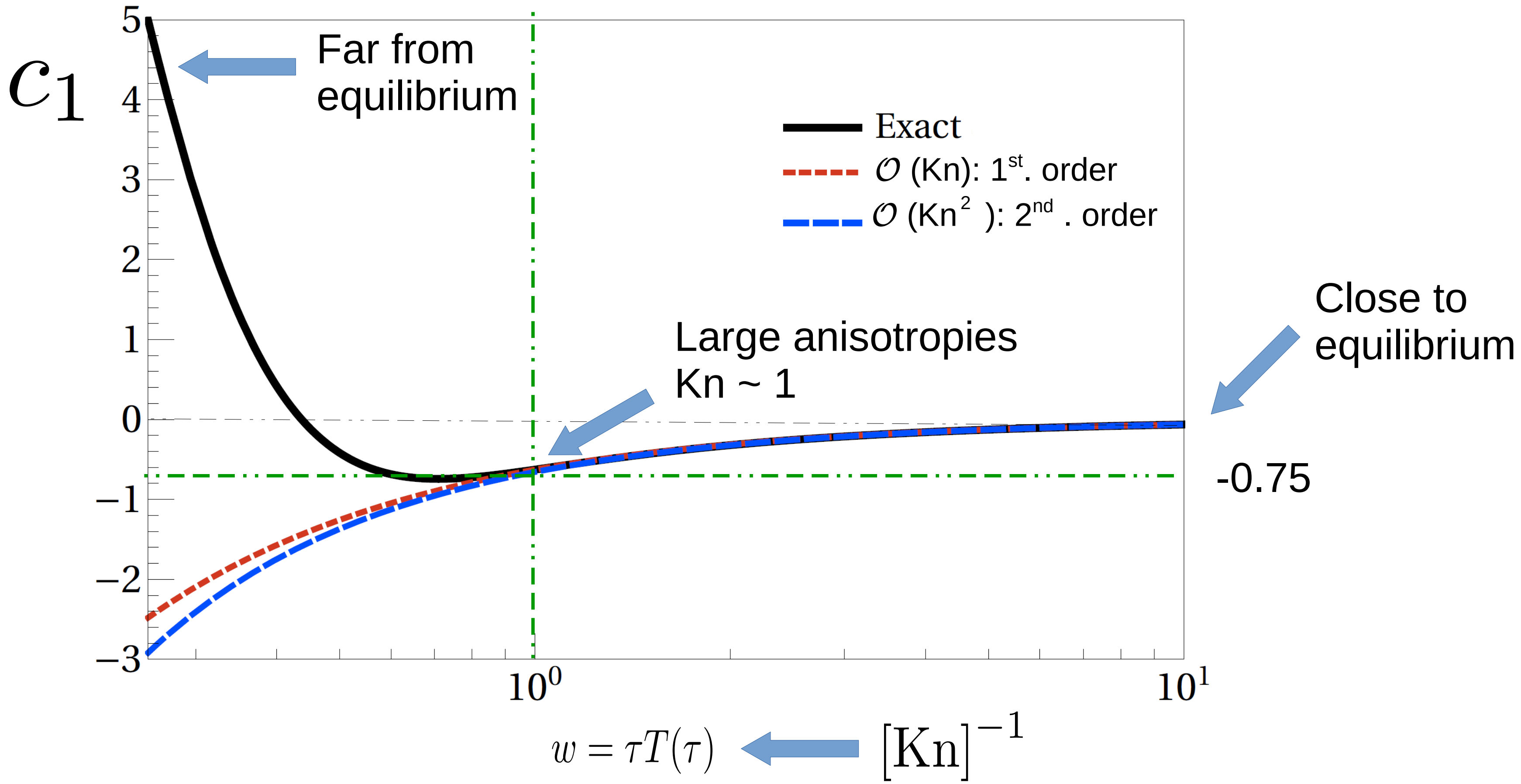


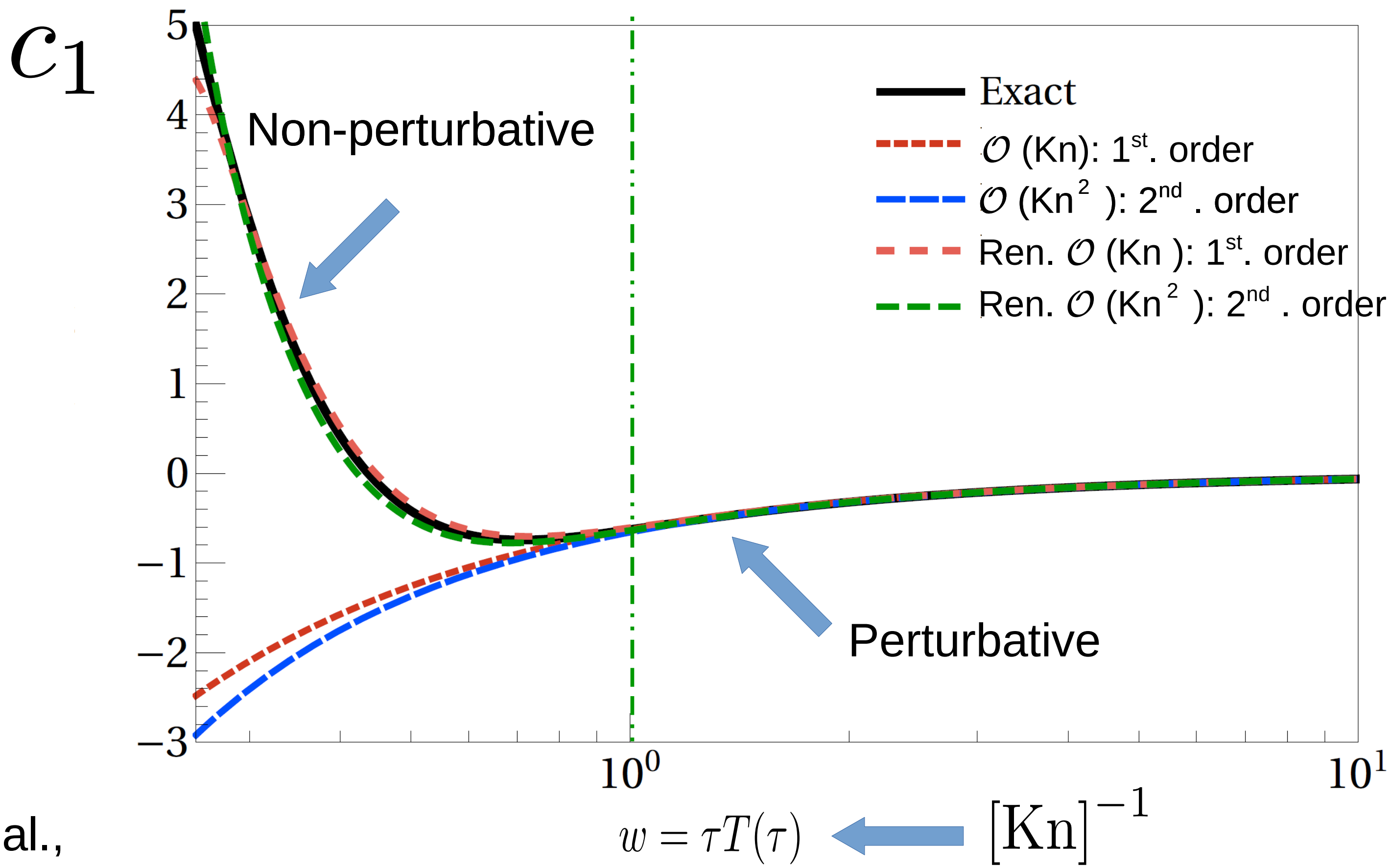
Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Resurgence and transseries

Asymptotic expansion → Transseries solutions
Costin (1998)

$$c_1 = \sum_{k=1}^{\infty} a_k \, [\mathrm{Kn}]^k \quad \longrightarrow \quad c_1 = \sum_{k=1}^{\infty} \left[a_k + \sum_{l=1}^{\infty} u_{k,l} \left(\sigma \, e^{-S/\mathrm{Kn}} \, [\mathrm{Kn}]^{\beta} \right)^{l} \right] [\mathrm{Kn}]^k$$

'Instanton'



Martinez et. al., (2018, 2019)

Transseries solutions to ODEs

If you have a non-linear differential equation of the form

$$\mathbf{y}' = \mathbf{f}_0(x) - \hat{\Lambda}\mathbf{y} - \frac{1}{x}\hat{B}\mathbf{y} + \mathbf{g}(x, \mathbf{y})$$

Then

$$\tilde{\mathbf{y}} = \tilde{\mathbf{y}}_0 + \sum_{\mathbf{k}\geq 0; |\mathbf{k}|>0} C_1^{k_1} \cdots C_n^{k_n} \mathrm{e}^{-(\mathbf{k}\cdot\boldsymbol{\lambda})x} x^{\mathbf{k}\cdot\mathbf{m}} \tilde{\mathbf{y}}_{\mathbf{k}}$$

$$\tilde{\mathbf{y}}_{\mathbf{k}} = x^{-\mathbf{k}(\boldsymbol{\beta}+\mathbf{m})} \sum_{l=0}^{\infty} \mathbf{a}_{\mathbf{k};l} x^{-l}$$

O. Coustin

1. Non-resonance condition: Λ *does not have null eigenvalues*
2. *Regularity when* $x \to \infty$

Duke Math. J. vol 93, No 2, 1998

How does this happen?

Linearize around the leading order term of the perturbative series

$$\frac{d\delta c_1}{dw} = \left.\frac{\partial F_1}{\partial c_1}\right|_{c_1=\bar{c}_1} \delta c_1$$

$$\delta c_1(w) = \sigma_1 \, e^{-S_1 w} \, w^{-b_1}$$

Lyapunov exponent (S_1)

Anomalous dimension (b_1)

Continue doing this procedure to all perturbative orders

Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Transseries solutions

$$c_1(w) = \left[u_{1,0}^{(1)}\,\sigma_1\,\zeta_1(w) + u_{1,0}^{(2)}\,[\sigma_1\,\zeta_1(w)]^2 + \cdots\right]$$
$$+ \frac{1}{w}\left[u_{1,1}^{(0)} + u_{1,1}^{(1)}\,\sigma_1\,\zeta_1(w) + u_{1,1}^{(2)}\,[\sigma_1\,\zeta_1(w)]^2 + \cdots\right]$$
$$+ \frac{1}{w^2}\left[u_{1,2}^{(0)} + u_{1,2}^{(1)}\,\sigma_1\,\zeta_1(w) + u_{1,2}^{(2)}\,[\sigma_1\,\zeta_1(w)]^2 + \cdots\right]$$

Perturbative IR data

Non-Perturbative Resummation of fluctuations around the IR perturbative expansion

Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Transport coefficients in the far from equilibrium regime

$$c_1(w) \equiv \sum_{k=0}^{+\infty} G_{1,k}(\sigma_1 \zeta_1(w)) w^{-k}$$

$$G_{1,k}(\sigma_1 \zeta(w)) = \sum_{n=0}^{\infty} u_{1,k}^{(n)} [\sigma_1 \, \zeta_1(w)]^n$$

Each function $G_{1,k}$ satisfies:

$$\lim_{w \to \infty} G_{1,k} = u_{1,k}^{(0)}$$ → Transport coefficient

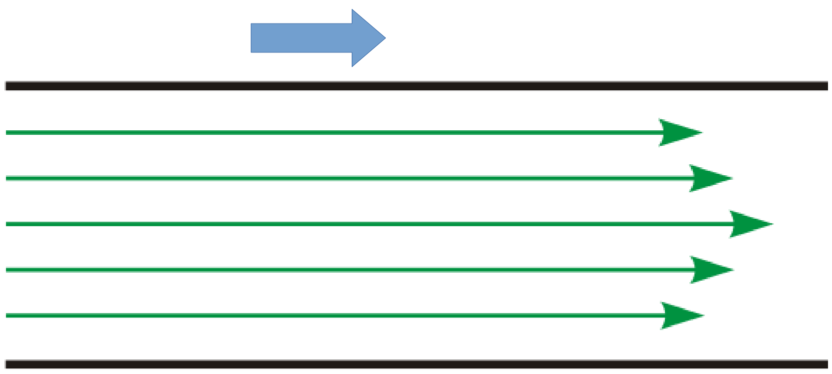
For instance

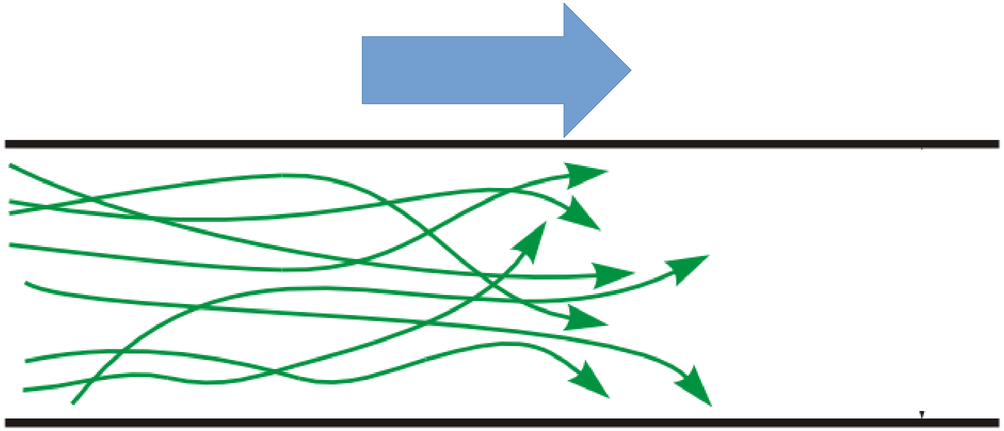
$$\frac{\eta}{s} = -\frac{3}{40} \lim_{w \to \infty} G_{1,1}(\sigma_1 \zeta(w))$$

⇒

$$\frac{\eta}{s}(w) = -\frac{3}{40} G_{1,k}(\sigma_1 \zeta(w))$$

Non-newtonian fluids and rheology





$$\pi_{yx} \sim \eta\, \partial_y v_x$$

$$\pi_{yx} \sim \eta(\partial_y v_x)\, \partial_y v_x$$

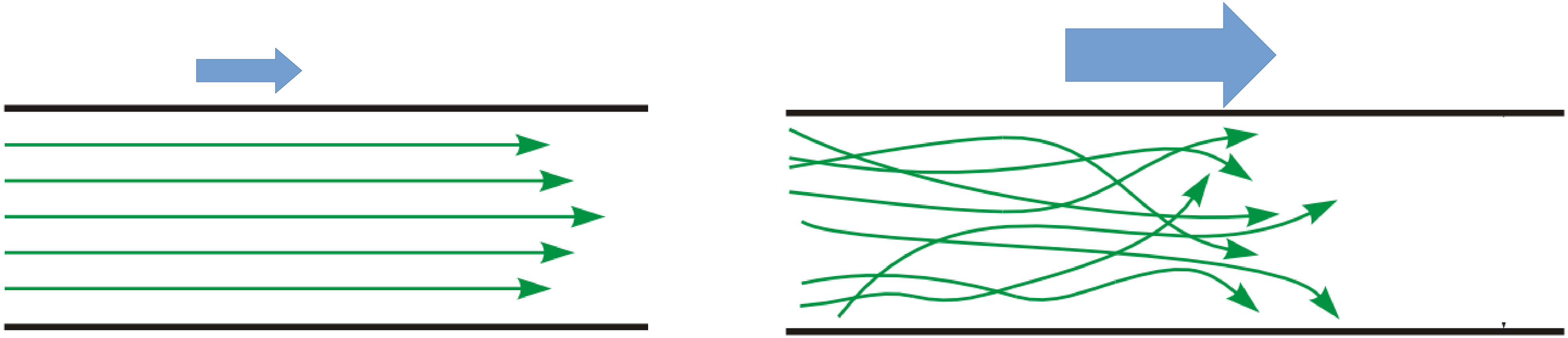
This is called shear thinning and shear thickening

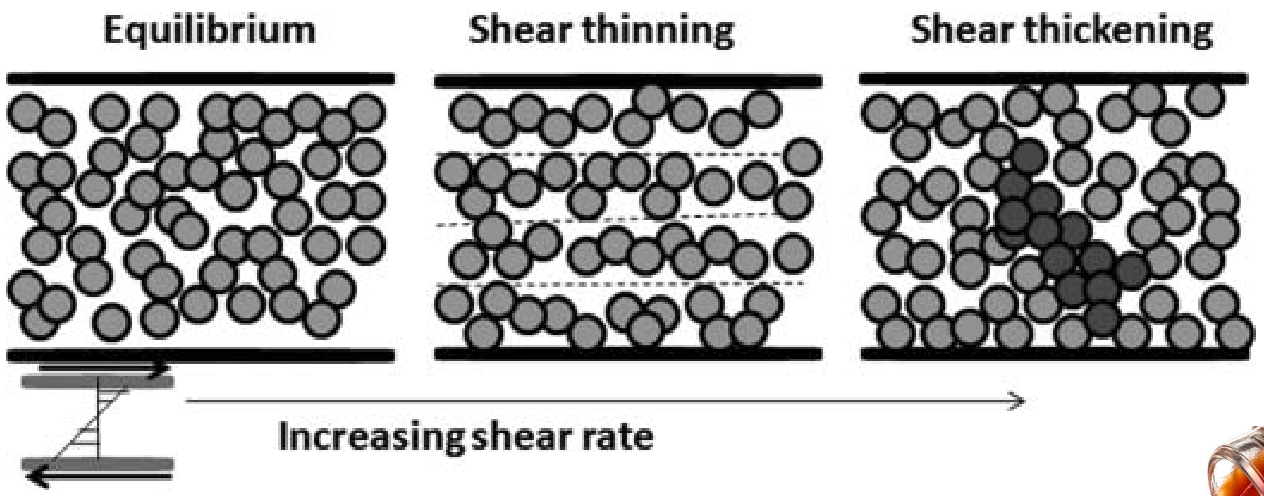
Shear viscosity

- Becomes a **function** of the **gradient of the flow velocity**
- can **increase** or **decrease depending** on the **size** of the **gradient of the flow velocity**

Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

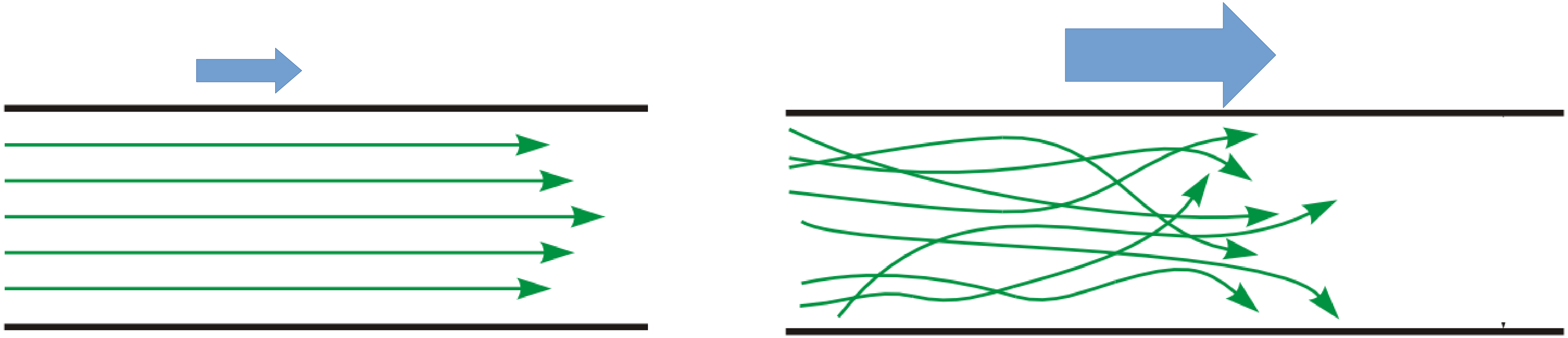
Non-newtonian fluids and rheology





Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Non-newtonian fluids and rheology



$$\frac{\eta}{s}(w) = -\frac{3}{40} G_{1,k}(\sigma_1 \zeta(w))$$

Thus, transseries solutions resummes non-perturbative contributions when the dissipative corrections are large. As a result, each transport coefficient is renormalized

Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Dynamical system as a RG flow

S. Gukov (2016)

RG flows are dynamical systems

Is it true in the other way around?

Sometimes a dynamical system is a RG flow.

- **Under which conditions?**

Dynamical system as a RG flow

Let's rewrite the ODEs in a precise manner

$$\frac{dc_1}{d\log w} = \beta_1(c_1, w)$$

Any observable $\mathfrak{O} = \mathfrak{O}(G_{1,k}(\sigma_1\zeta_1))$

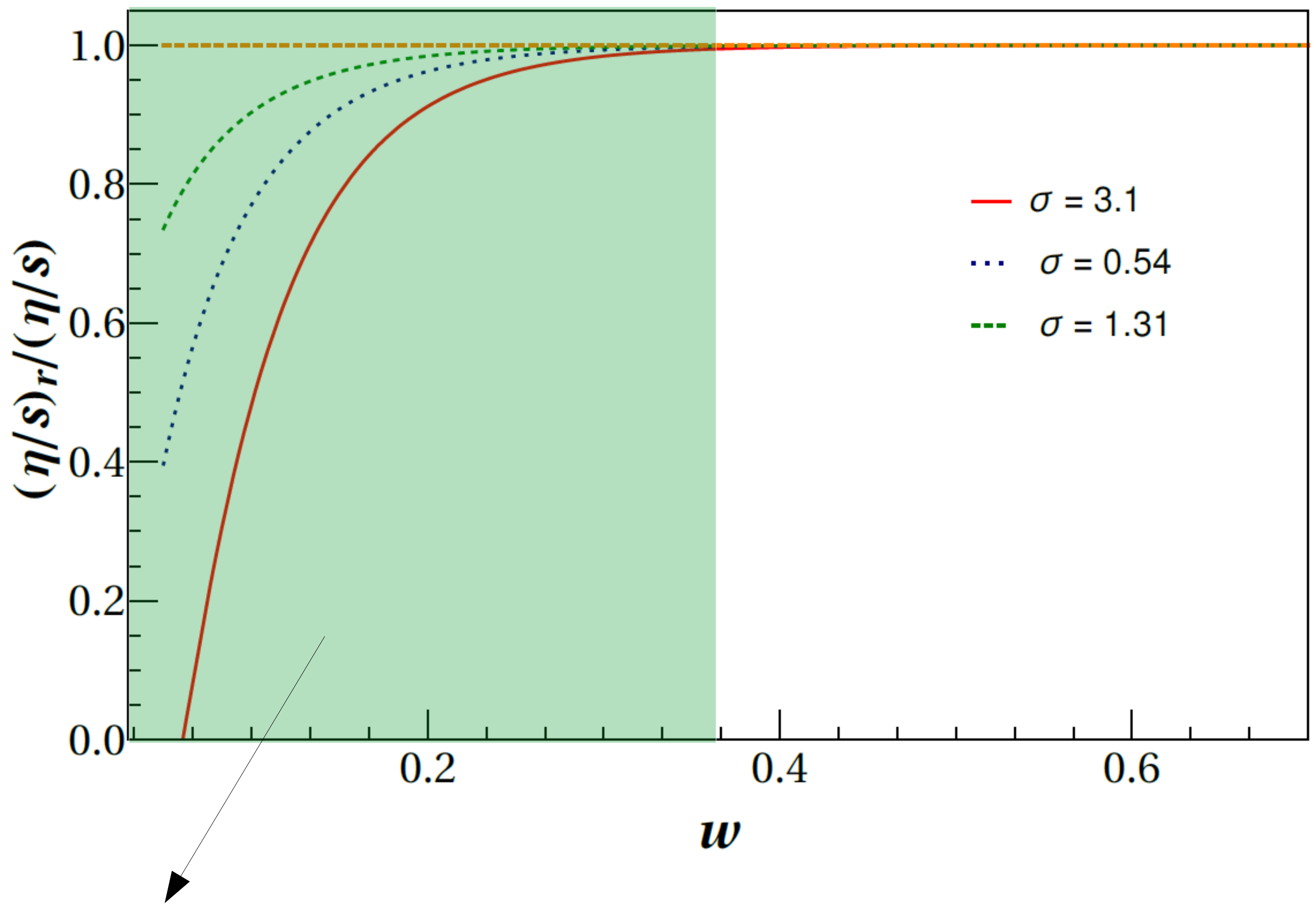
$$\frac{d\mathfrak{O}(G_{1,k}(\sigma_1\zeta_1))}{d\log w} = -\sum_{k=0}^{\infty}\left[(b_1 + S_1 w)\,\hat{\zeta}_1 G_{1,k}(\sigma_1\zeta_1)\right]\frac{\partial\mathfrak{O}}{\partial G_{1,k}}$$

RG flow equation for shear viscosity over entropy ratio is simply obtained by using

$$\frac{\eta}{s}(w) = -\frac{3}{40}G_{1,k}(\sigma_1\zeta(w))$$

Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Transient rheological behavior



Shear thickening

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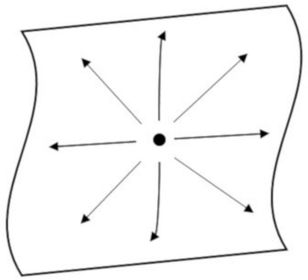
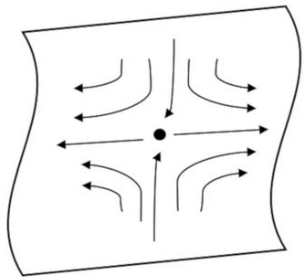
UV regime





UV expansion

Let's start by following a similar procedure by changing w ⇨1/z and expand when z⇨∞

$$\frac{dc_1}{dz} = F_1(c_1, z)$$

Perturbative solutions

$$c_1(z) = \sum_{k=1}^{\infty} \frac{v_{1,k}}{z^k}$$

Linearized perturbations

$$\delta c_1^{\pm}(z) = \mu_1^{\pm} \, z^{\alpha_1^{\pm}}$$

Power law behavior

Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

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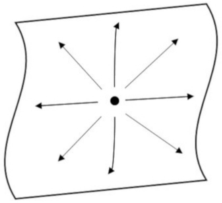
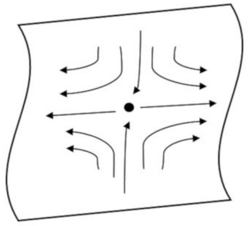


Fast decay



Growth

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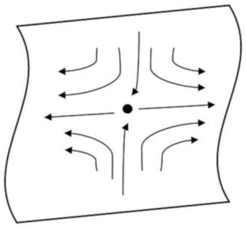
UV expansion

Consider the expansion around saddle point



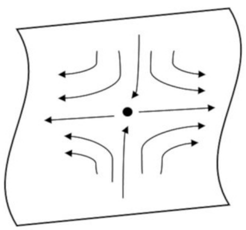
$$c_1(z) = \sum_{k=1}^{\infty} \frac{v_{1,k}}{z^k}$$

Power law series:
divergent
If truncated one can extend its radius of convergence by analytically continuing!!

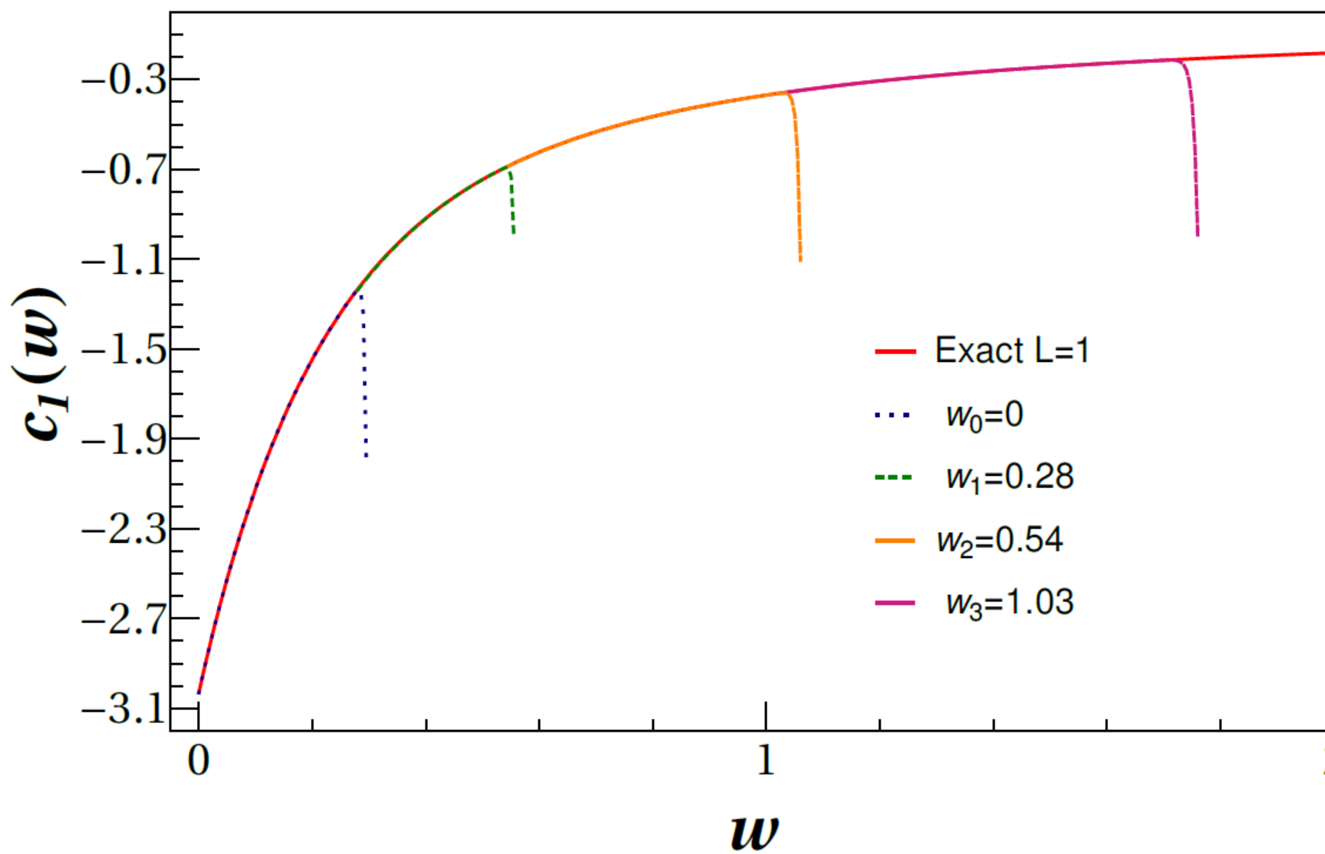
Martinez et. al.   1805.07881, 1901.08632, 1911.06406, 2011.08235

UV expansion

Consider the expansion around saddle point



$$c_1(z) = \sum_{k=1}^{\infty} \frac{v_{1,k}}{z^k}$$



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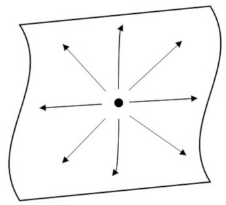
UV expansion

Consider the expansion around source point



$$c_1 = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} v_{1,k}^{(n)} \varphi_k^n$$

$$\varphi_k^n := z^{-k} [\xi_1(z)]^n \,, \qquad \xi_1(z) = \mu_1 \, z^{\alpha_1} \,.$$

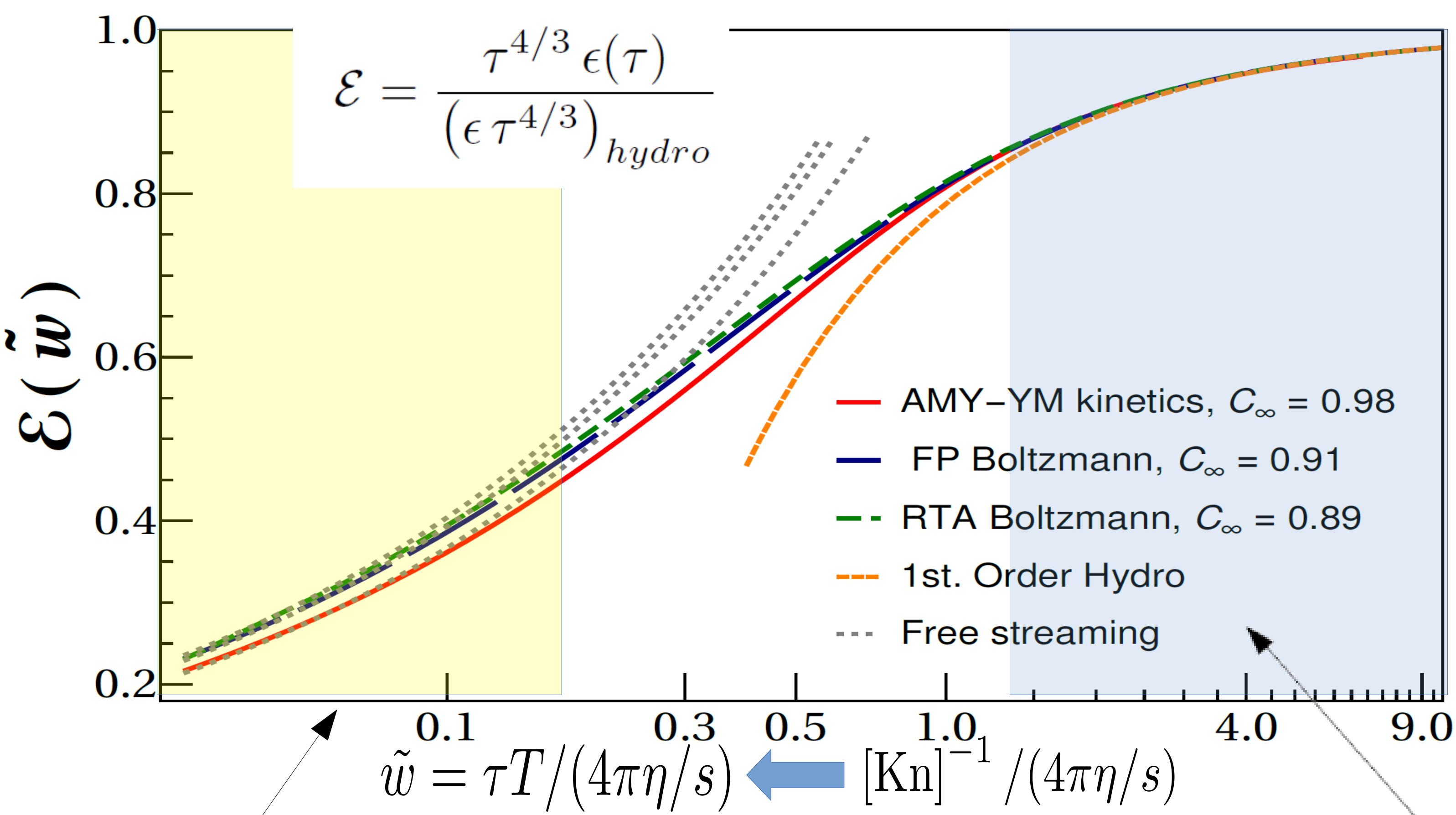
Perturbative

Power law decay
No Instantons

Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Universal properties

Universal features of attractors



Early-time free-streaming

Late-time hydrodynamical behavior

Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Conclusions

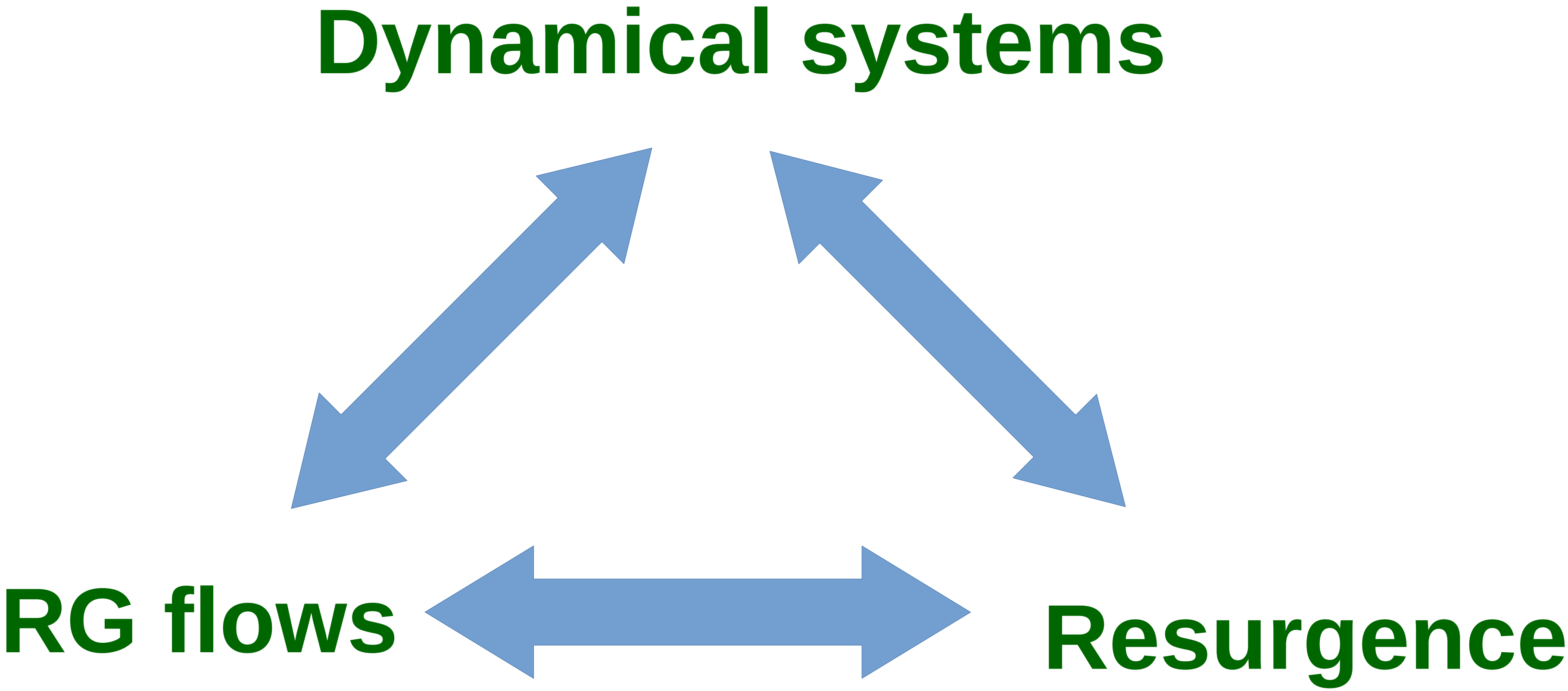
1. UV and IR expansions present different behavior for the Bjorken flow

2. IR solutions are written as a multiparameter transseries

- Transport coefficients get effectively renormalized after resumming non-perturbative instanton-like contributions

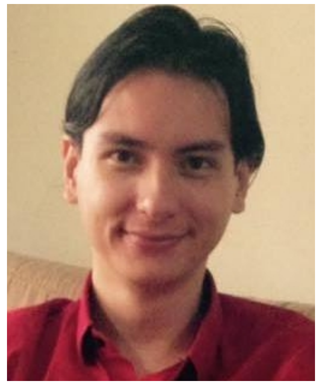
3. UV expansions present power law solutions with a finite radius of convergence

4. Early and late time behavior are determined by free streaming and viscous hydrodynamics respectively.


Dynamical systems
RG flows
Resurgence

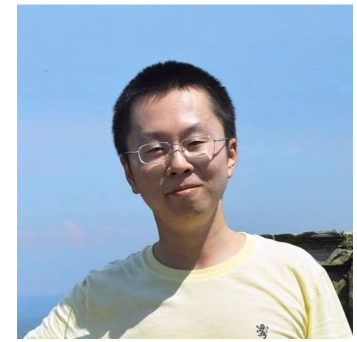
Excellent group of collaborators

A. Behtash



C. N. Camacho

S. Kamata



H. Shi

J. Jankowski

T. Schaefer

V. Skokov

M. Spalinski

Outlook

- Resurgence analysis of other relevant systems

  1. Jet quenching
  2. Cosmology
  3. Cold atoms

- Challenges:
  1. How to generalize to arbitrarily expanding geometries?
  2. Phase transitions?
  3. Effective action (Lyapunov functionals)

  For Gubser flow: Behtash. et. al. PRD 97 044041 (2018)

Backup slides

Asymptotics in the Boltzmann equation

Usually the distribution function is expanded as series in Kn, i.e.,

$$f(x^\mu, p) = \sum_{k=0}^{\infty} (Kn)^k \, f_k(x^\mu, p) \qquad Kn \equiv \frac{l}{L}$$

Macroscopic quantities are simply averages , e.g.,

$$T^{\mu\nu} = \int_{\mathbf{p}} p^\mu p^\nu f(x^\mu, \mathbf{p}) \quad \Rightarrow \quad T^{\mu\nu} = \sum_{k=0}^{\infty} (Kn)^k T_k^{\mu\nu}$$

$$T_0^{\mu\nu} = (\epsilon + p(\epsilon))\, u^\mu u^\nu + p(\epsilon) g^{\mu\nu}$$ → Ideal fluid $\mathcal{O}(\mathrm{Kn}^0)$

$$T_1^{\mu\nu} = -\eta\, \sigma^{\mu\nu}$$ → $\mathcal{O}(\mathrm{Kn})$: Navier-Stokes

$$T_2^{\mu\nu}$$ → $\mathcal{O}(\mathrm{Kn}^2)$: IS, etc

Asymptotics in the Boltzmann equation

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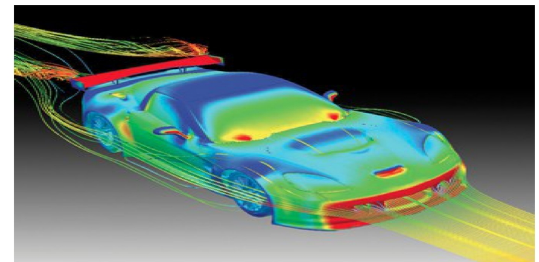
$$f(x^\mu, p) = \sum_{k=0}^{\infty} (Kn)^k \, f_k(x^\mu, p) \qquad Kn \equiv \frac{l}{L}$$

Microscopic scale (Mean free path) $\quad l \sim \lambda_{mfp}$

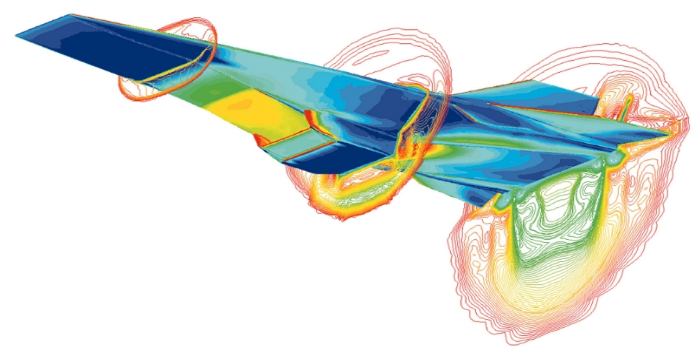
Macroscopic scale (spatial gradients) $\quad \frac{1}{L} \sim \partial_i v^i$



$L \sim 1\, m \qquad \ell \sim 10^{-7}\, m$

Expansion fails if

$$Kn \sim \frac{l}{L} \sim 1$$



High-altitude flights

Dryden Flight Research Center ED97 43968-01
HYPER-X AT MACH 7: This computational fluid dynamic (CFD) image is of the Hyper-X vehicle at the Mach 7 test condition with the engine operating.