IJS

University of Ljubljana
Faculty of Mathematics and Physics

False Vacuum Decay

Miha Nemevšek

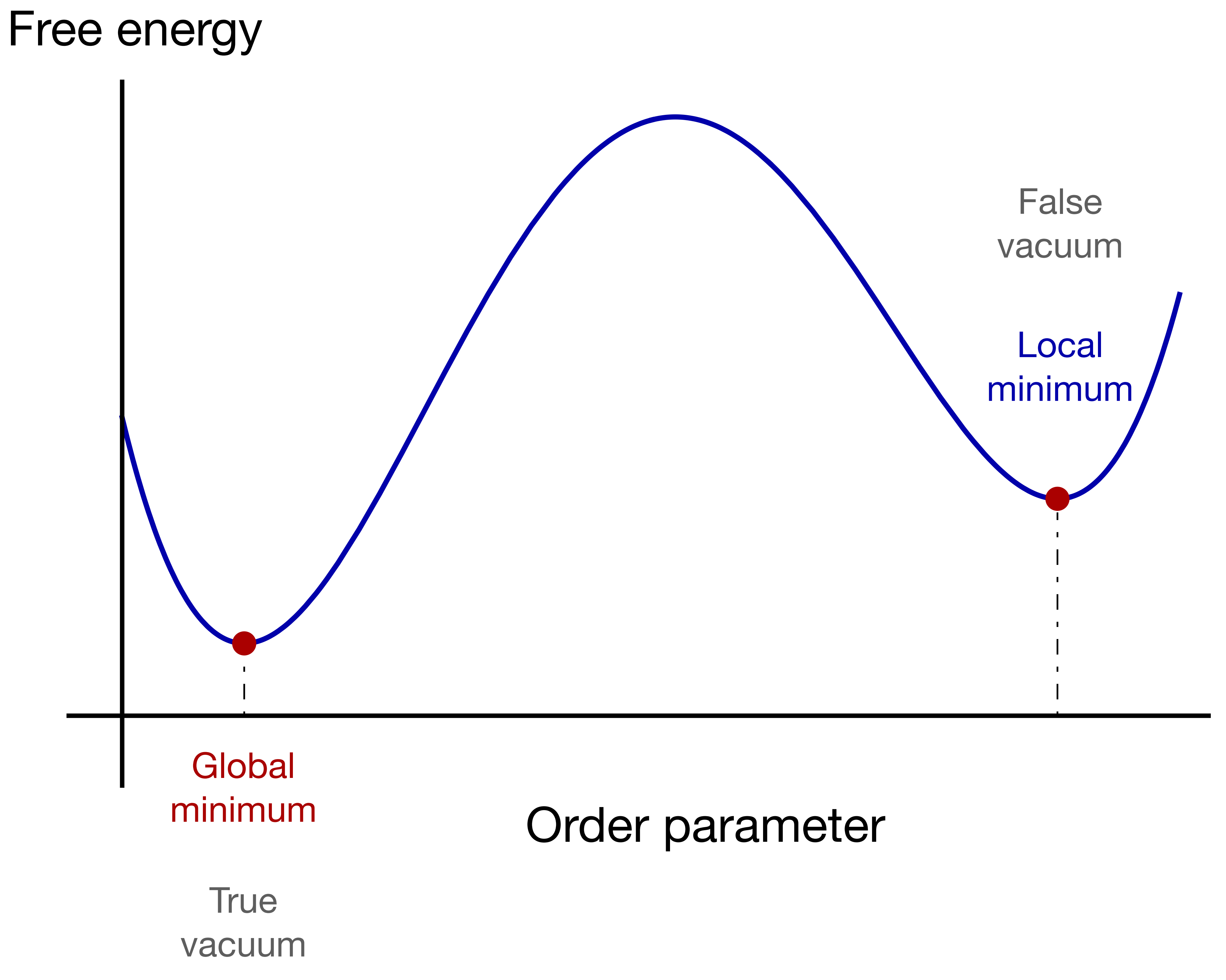
with Victor Guada (IJS), Alessio Maiezza (IRB), Matevž Pintar (Gen-i)

based on: 1803.02227 PRD | 2002.00881 Phys. Comput. Comm | 2009.01535 PRD

Holography, strings and transport seminar

Ljubljana 2020

Introduction


Free energy
False
vacuum
Local
minimum
Global
minimum
Order parameter
True
vacuum

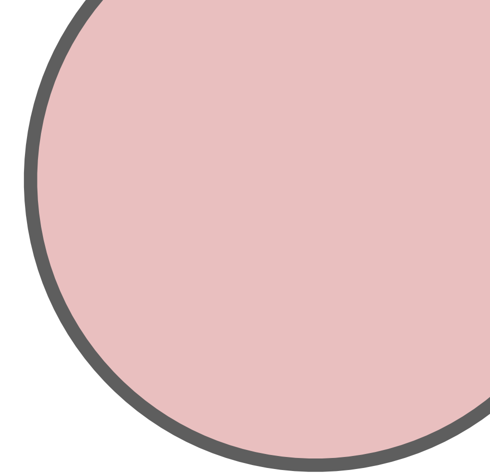
First order phase transition

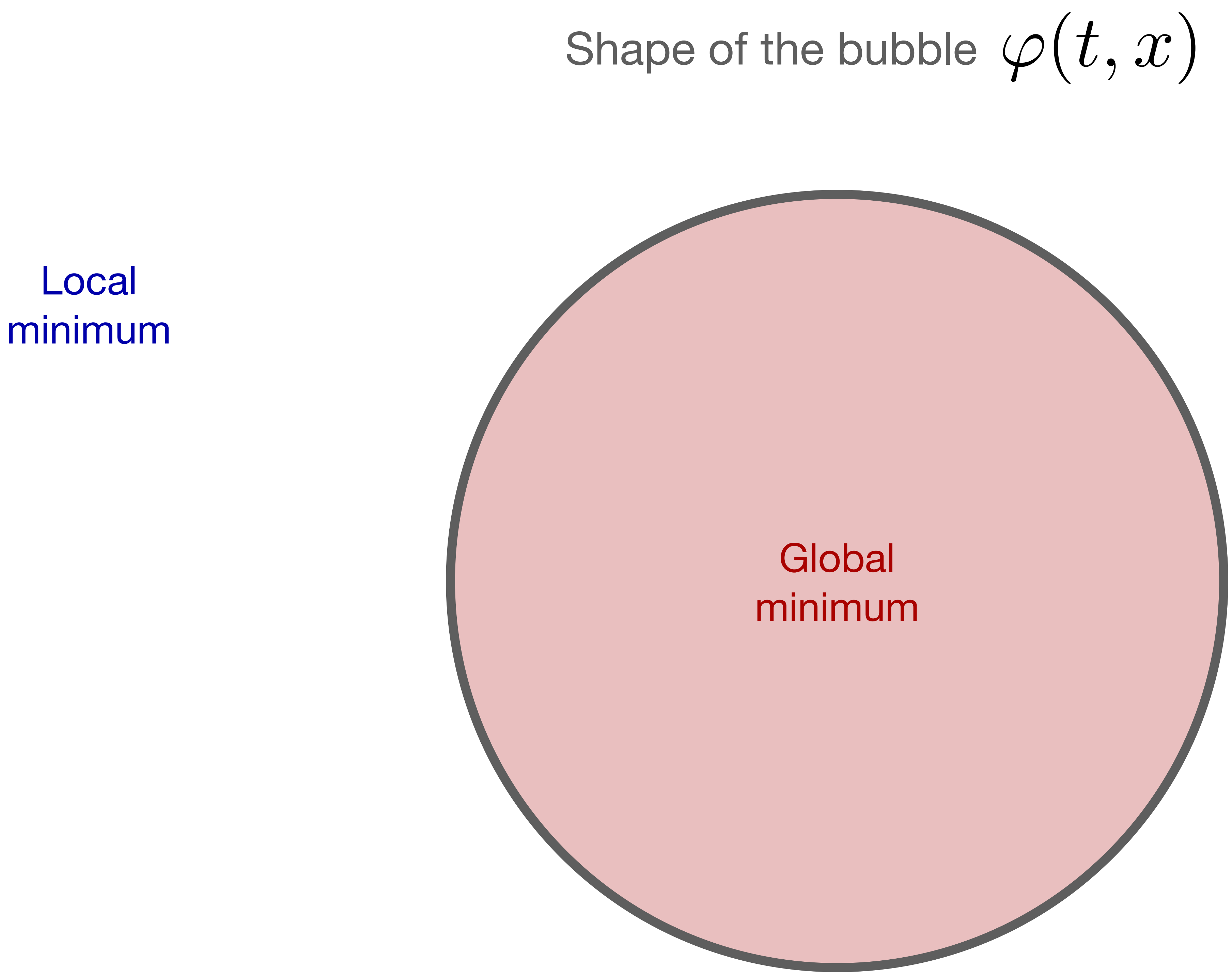
Decay rate

$$\Gamma = A\, e^{-\mathcal{S}_0}$$

Local
minimum

First order phase transition





Physical motivation

Vacuum stability

SM lifetime - zero and finite T ($m_t,\ \alpha_s$)

BSM stability - mass spectra and quartics,
selection of vacuua (landscape)

Phase transitions in the early universe

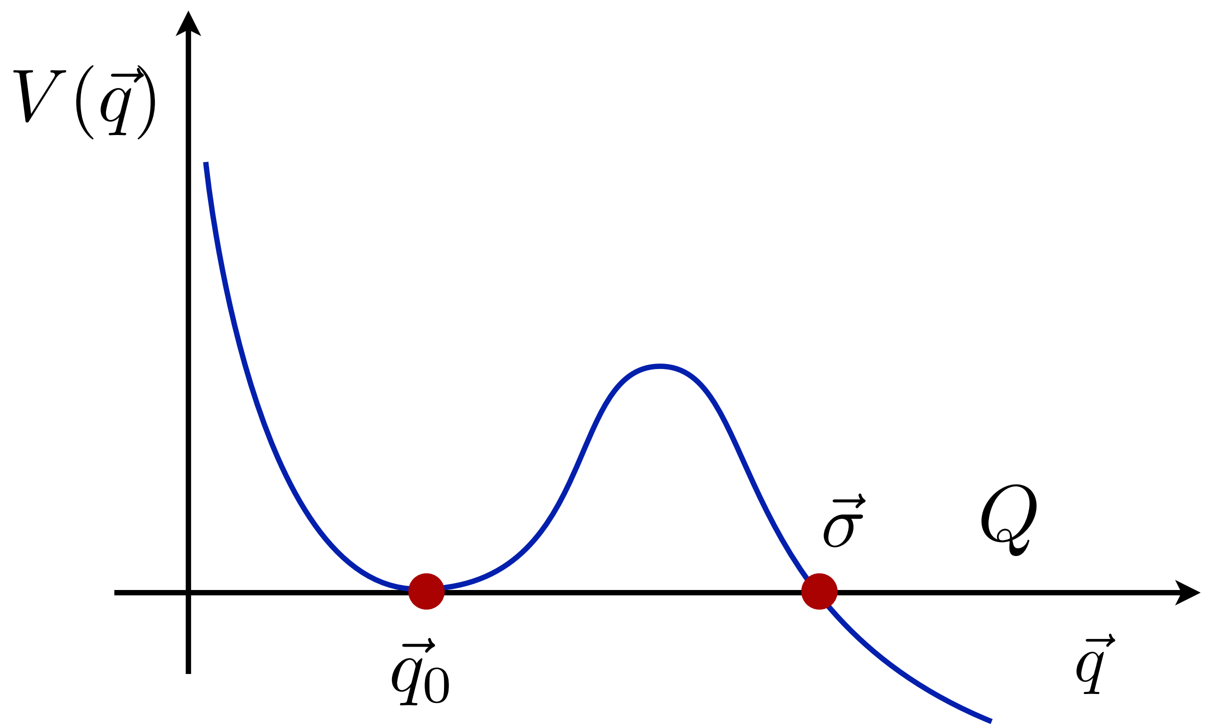
Gravitational wave production, primordial *B*-fields

Baryogenesis / bubble wall fermion conversion

Inflation, dark energy

Quantum mechanics textbook case is alpha decay

$$L = \frac{1}{2}\dot{q}^2 - V(q)$$



WKB '26

barrier penetration

$$B = 2\int_{q_0}^{\sigma} \mathrm{d}q \sqrt{2V(q)}$$

Prefactor: dimensionful, estimated from the kinetic energy of the outgoing alpha

$$Q = \frac{m_{\mathrm{He}} v^2}{2} \qquad \tau_c^{-1} = \sqrt{\frac{2Q}{m_{\mathrm{He}} a}}$$

$$\frac{\Gamma}{\mathcal{V}} = \tau_c^{-1}\, e^{-B}$$

Quantum field theory real scalar with two minima

Coleman '77

$$\mathcal{L} = \frac{1}{2}\partial\varphi^2 - V(\varphi) \qquad \Rightarrow \qquad \mathcal{L}_E = \frac{1}{2}\partial_E\varphi^2 + V(\varphi)$$

$$\frac{\Gamma}{\mathcal{V}} = \int \mathcal{D}\varphi\, e^{-S[\varphi]}$$

$$S[\varphi] \simeq S[\overline{\varphi}] + \underbrace{\left.\frac{\delta S}{\delta\varphi}\right|_{\overline{\varphi}}}_{=0} + \frac{1}{2}\left.\frac{\delta^2 S}{\delta\varphi^2}\right|_{\overline{\varphi}} + \dots$$

bounce extremize fluctuations

$O(4)$ Coleman, Glaser, Martin '78

$$\rho^2 = t^2 + \sum x_i^2$$

Euclidean time = radius of the bubble

Quantum field theory real scalar with two minima

Coleman '77

$$\mathcal{L} = \frac{1}{2}\partial\varphi^2 - V(\varphi) \qquad \Rightarrow \qquad \mathcal{L}_E = \frac{1}{2}\partial_E\varphi^2 + V(\varphi)$$

$$\frac{\Gamma}{\mathcal{V}} = \int \mathcal{D}\varphi \, e^{-S[\varphi]}$$

$$S[\varphi] \simeq S[\overline{\varphi}] + \underbrace{\left.\frac{\delta S}{\delta\varphi}\right|_{\overline{\varphi}}}_{=0} + \frac{1}{2}\left.\frac{\delta^2 S}{\delta\varphi^2}\right|_{\overline{\varphi}} + \dots$$

"...there always exists an *O(4)*-invariant bounce and it always has strictly lower action than any non-*O(4)* invariant bounce. The rigor of our proof is matched only by its tedium; I wouldn't lecture on it to my worst enemy." Coleman, Erice lectures '77

multi-fields

Blum, Honda, Sato, Takimoto, Tobioka '16

The bounce D dimensional symmetric Euclidean action

$$S_D = \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \int_0^\infty \rho^{D-1}\, \mathrm{d}\rho \left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right)$$

D = *4* : FV decay at *T* = *0* — Coleman '77

D = *3* : FV nucleation at finite *T* — Affleck '81, Linde '83

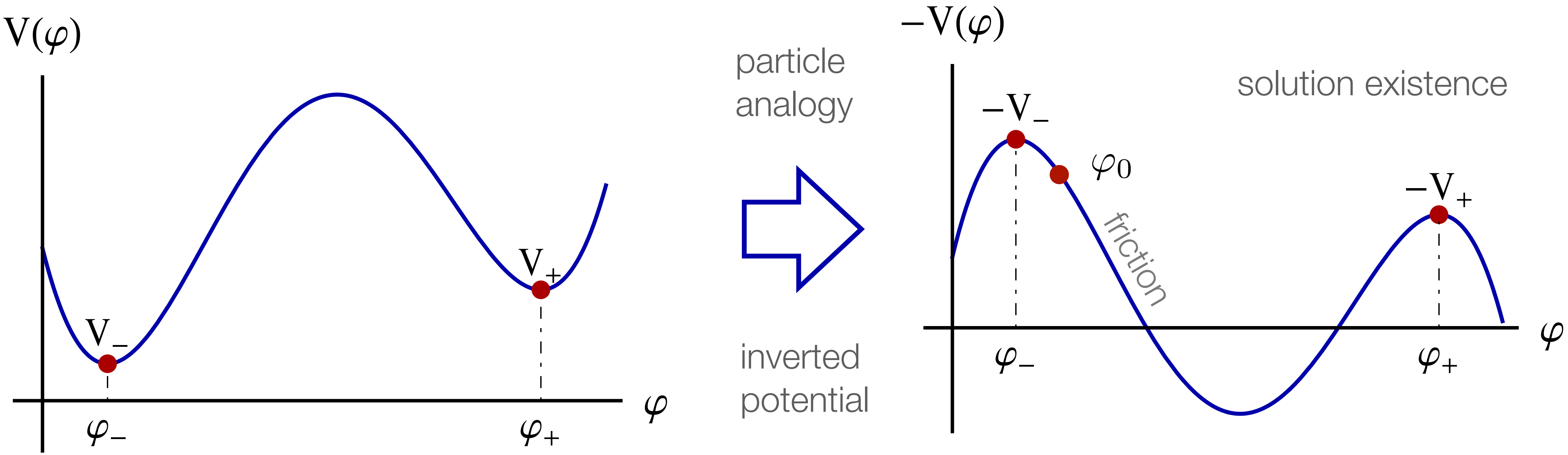
bounce equation

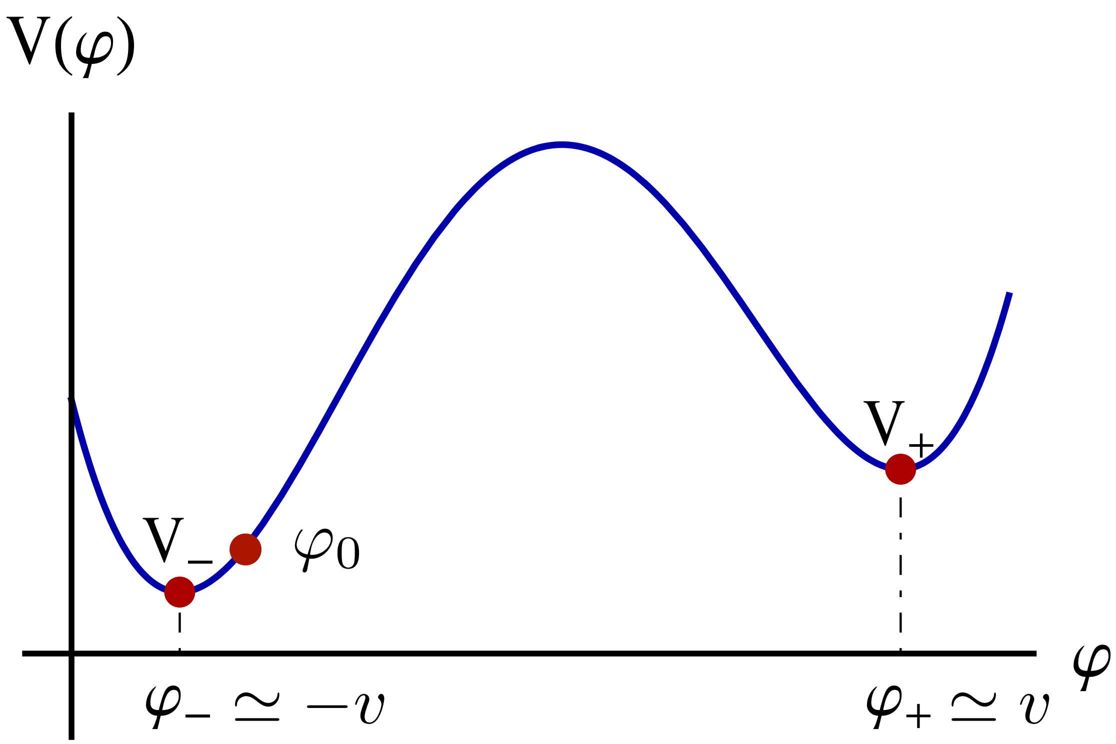
$$\ddot{\varphi} + \frac{D-1}{\rho}\dot{\varphi} = dV$$

friction

boundary conditions

$$\varphi(0) = \varphi_0, \quad \varphi(\infty) = \varphi_+, \quad \dot{\varphi}(0,\infty) = 0$$





Thin wall approximation

Coleman '77

$$V(\varphi) = \frac{\lambda}{8}\left(\varphi^2 - v^2\right)^2 + \varepsilon\left(\frac{\varphi - v}{2v}\right), \qquad S_1 = \frac{v^3\sqrt{\lambda}}{3}$$

small ε limit $\quad \varphi_0 \simeq \varphi_-$ $\quad$ until $\quad \rho = R$

field solution

$$\varphi(\rho) = \begin{cases} -v, & \rho \ll R \\ \varphi_1(\rho - R), & \rho \approx R \\ v, & \rho \gg R \end{cases} \qquad \varphi_1(\rho) = v \tanh\left(\frac{\sqrt{\lambda}v}{2}\rho\right)$$

extremize the action

bounce action

$$S_E = 2\pi^2 \int_0^\infty \rho^3 d\rho \left(\frac{1}{2}\dot{\varphi}^2 + V\right)$$

$$= -\frac{1}{2}\pi^2 R^4 \varepsilon + \pi^2 R^3 S_1$$

volume $\qquad$ surface

$$\frac{dS_E}{dR} = 0 \quad \Rightarrow \quad R = \frac{3S_1}{\varepsilon}$$

$$S_E = \frac{27\pi^2}{2}\frac{S_1^4}{\varepsilon^3}$$

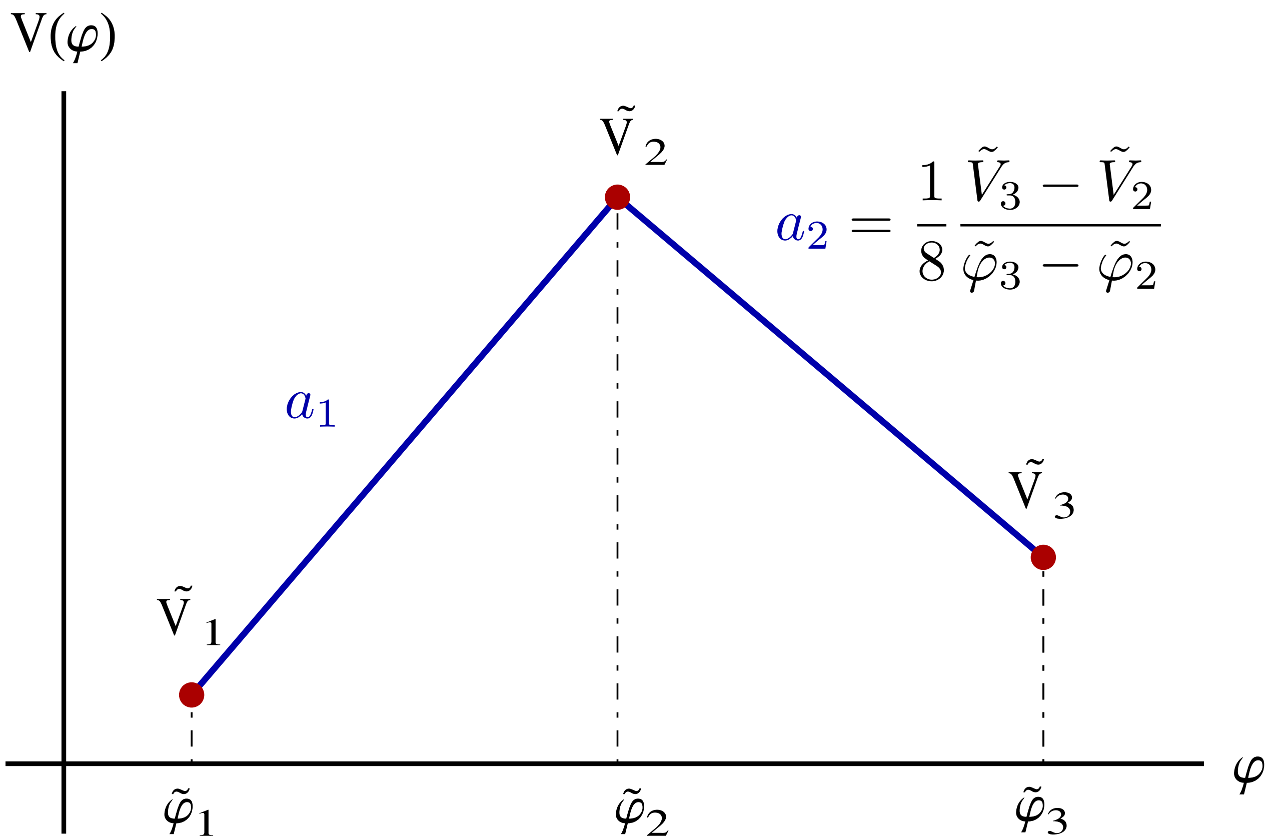
runaway

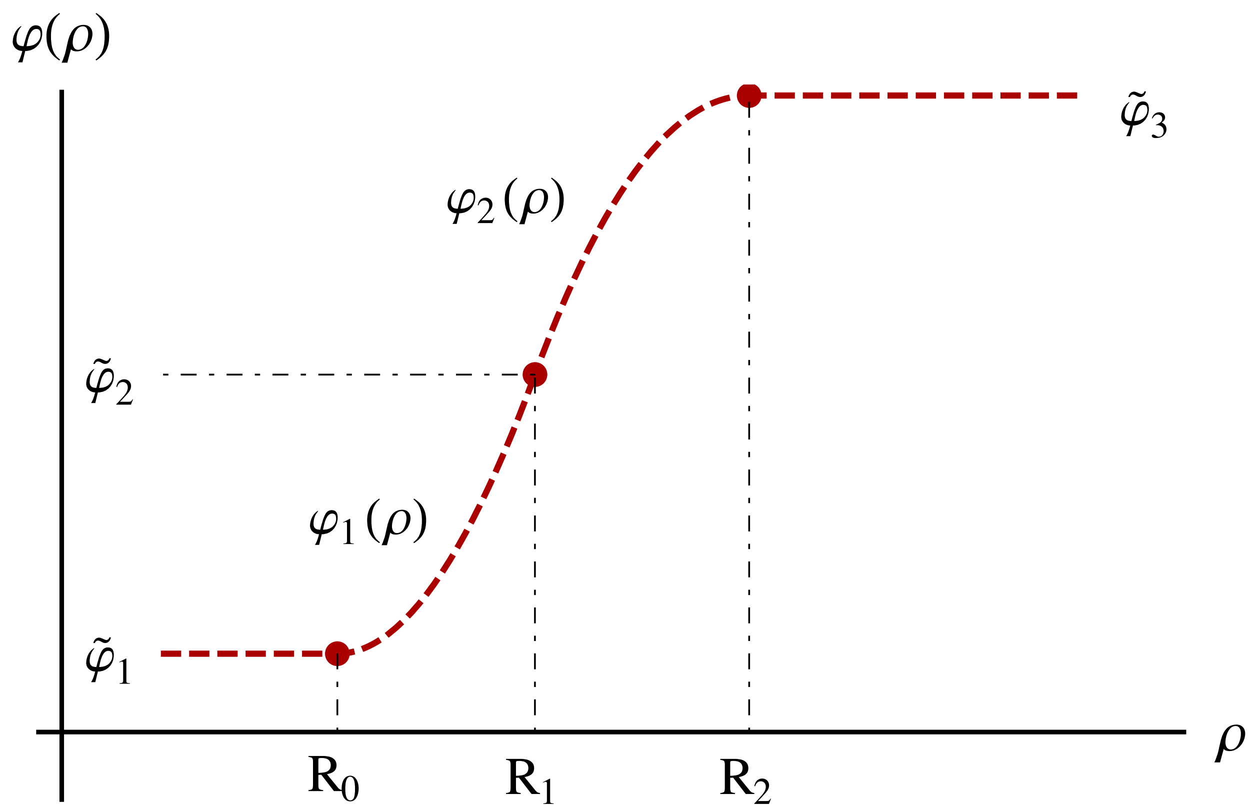
$$\frac{d^2 S_E}{dR^2} < 0$$

Coleman '77

Bödeker, Moore '09, '17

Triangular





Linear potentials

Duncan, Jensen '92

- triangle and box

Exact solution

$$\ddot{\varphi} + \frac{3}{\rho}\dot{\varphi} = dV = 8\,a$$

$$\varphi = v + a\,\rho^2 + \frac{b}{\rho^2}$$

Initial conditions @ R_0

- a) $\varphi_1(0) = \varphi_0, \quad \dot{\varphi}_1(0) = 0$

  $v_1 = \varphi_0, \quad b_1 = 0$

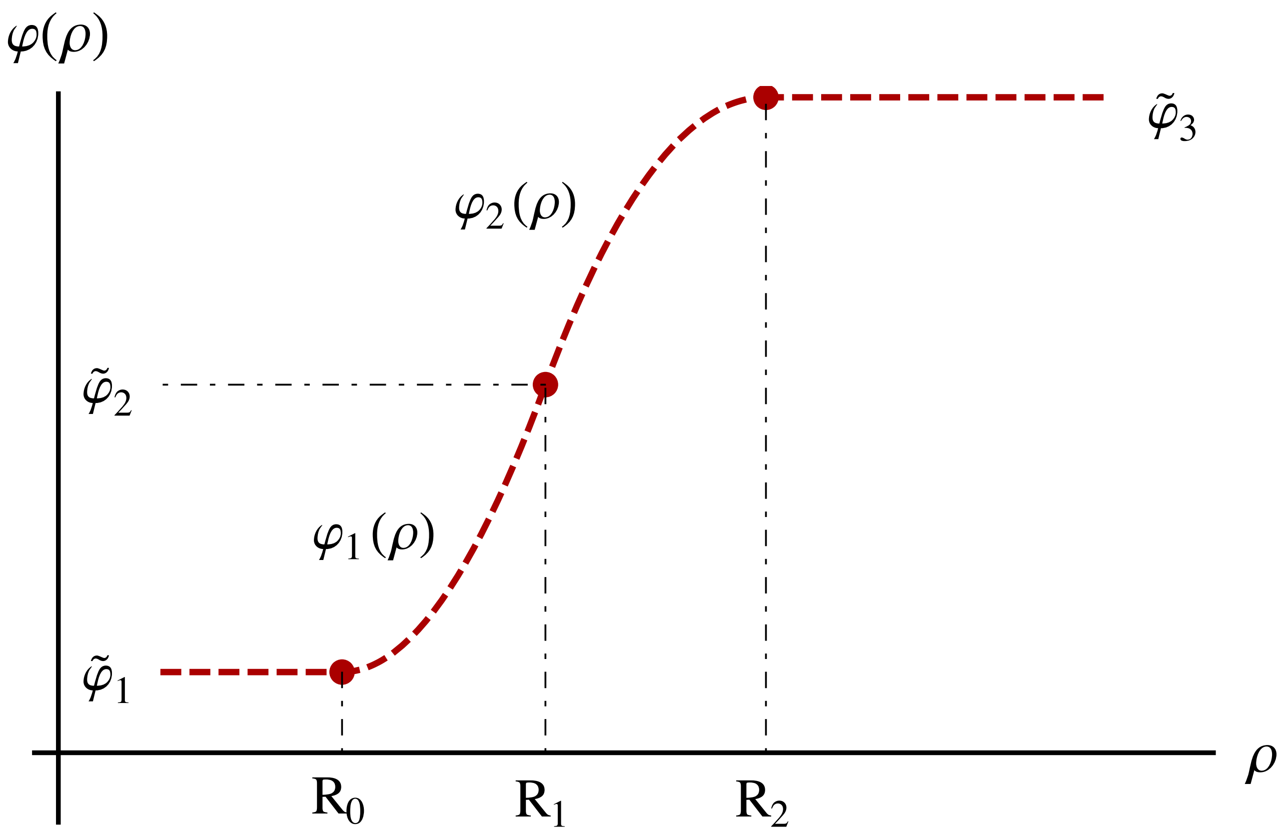
  shoot in φ_0

- b) $\varphi_1(R_0) = \tilde{\varphi}_1, \quad \dot{\varphi}_1(R_0) = 0$

  or R_0

  $v_1 = \tilde{\varphi}_1 - 2\,a_1\,R_0^2, \quad b_1 = a_1\,R_0^4$

Triangular



Matching conditions @ R_1

$$\varphi_1(R_1) = \varphi_2(R_1) = \tilde{\varphi}_2, \quad \dot{\varphi}_1(R_1) = \dot{\varphi}_2(R_1)$$

Final conditions @ R_2

$$\varphi_2(R_2) = \tilde{\varphi}_3, \quad \dot{\varphi}_2(R_2) = 0$$

$$v_2 = \tilde{\varphi}_3 - 2\, a_2\, R_2^2, \quad b_2 = a_2\, R_2^4$$

Complete solution - a) works in D-dimensions

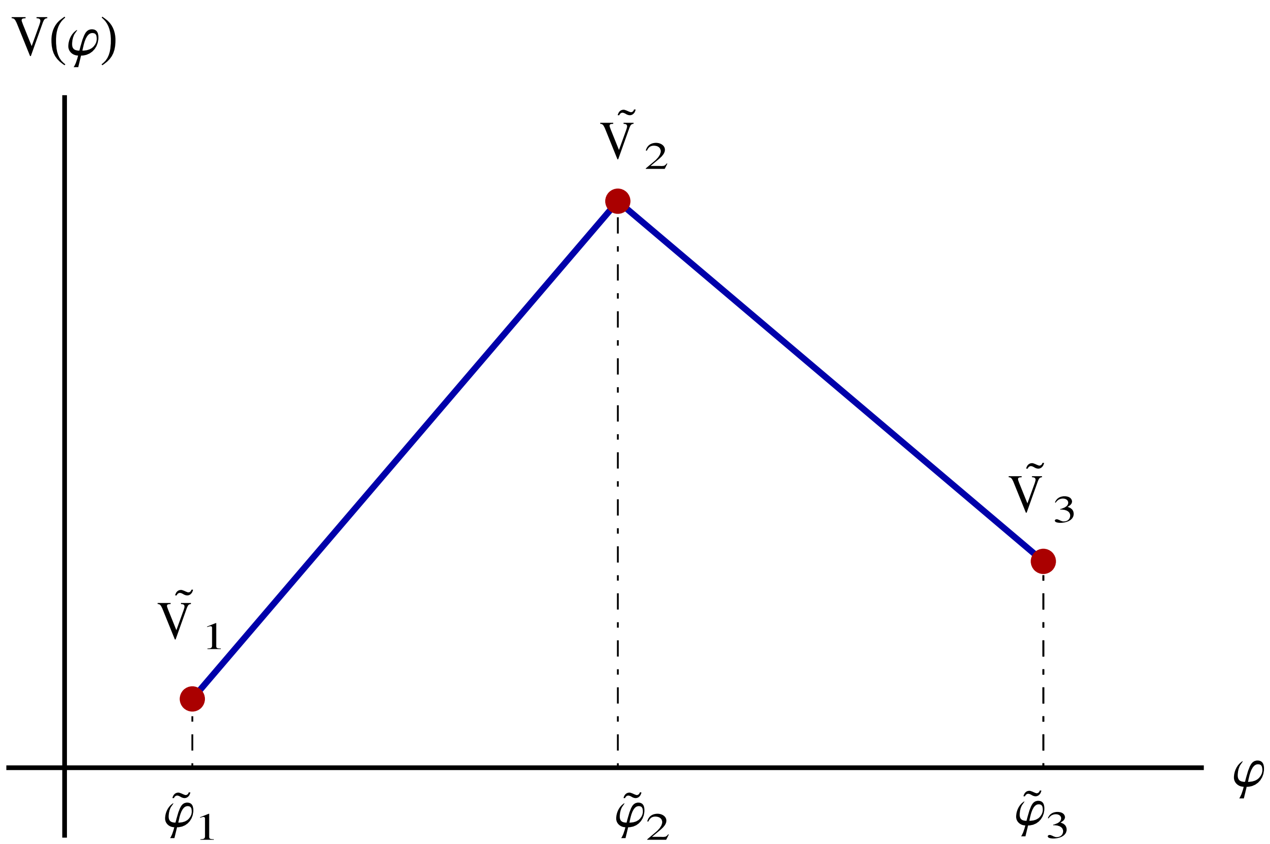
- a) $\varphi_0 = \dfrac{\tilde{\varphi}_3 + c\,\tilde{\varphi}_2}{1+c}, \qquad c = 2\dfrac{a_2 - a_1}{a_1}\left(1 - \sqrt{\dfrac{a_2}{a_2 - a_1}}\right) \qquad R_1 = \sqrt{\dfrac{D}{4}\left(\dfrac{\tilde{\varphi}_2 - \varphi_0}{a_1}\right)}$

- b) $R_1 = \dfrac{1}{2}\dfrac{\tilde{\varphi}_3 - \tilde{\varphi}_1}{\sqrt{a_1\,(\tilde{\varphi}_2 - \tilde{\varphi}_1)} - \sqrt{-a_2\,(\tilde{\varphi}_3 - \tilde{\varphi}_2)}}$

$$R_0^2 = R_1\left(R_1 - \sqrt{\frac{\tilde{\varphi}_2 - \tilde{\varphi}_1}{a_1}}\right)$$

$$R_2^2 = R_1\left(R_1 + \sqrt{\frac{\tilde{\varphi}_3 - \tilde{\varphi}_2}{-a_2}}\right)$$

Summary



Complete exact analytic solution

Solved in terms of Euclidean radius

Stable in thin wall, goes over to TWA

with limited validity

Dutta, Hector, Konstandin,
Vaudrevange, Westphal '12

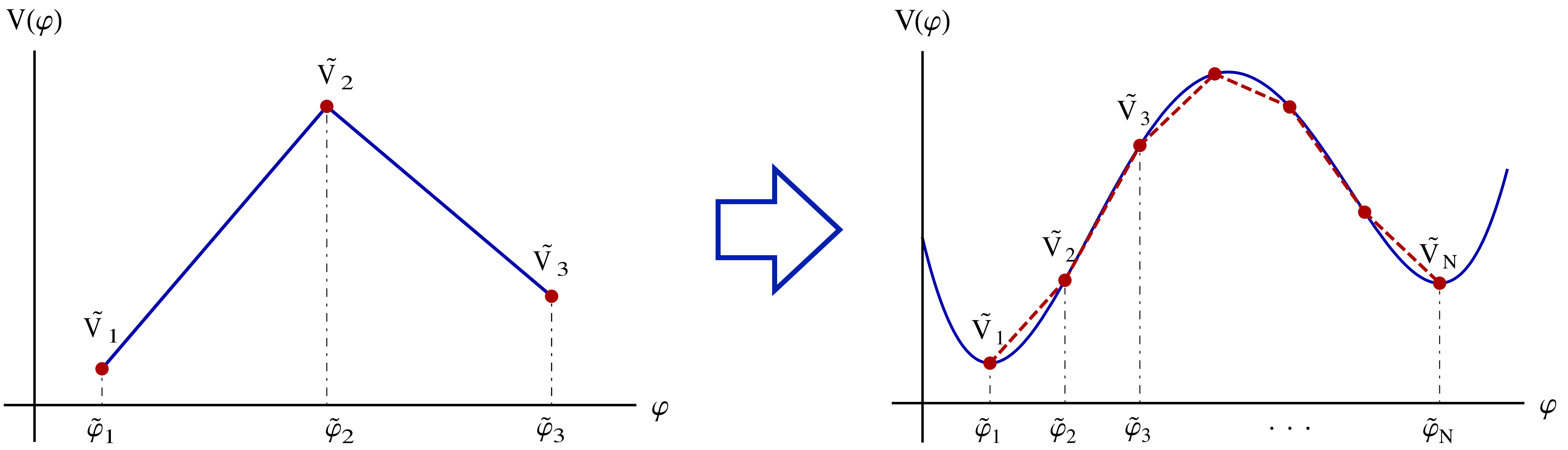
Analytic continuation in Minkowski space

describes the bubble evolution Pastras '11

Polygonal bounce

Polygonal bounces

Extend to more segments and *D* dimensions

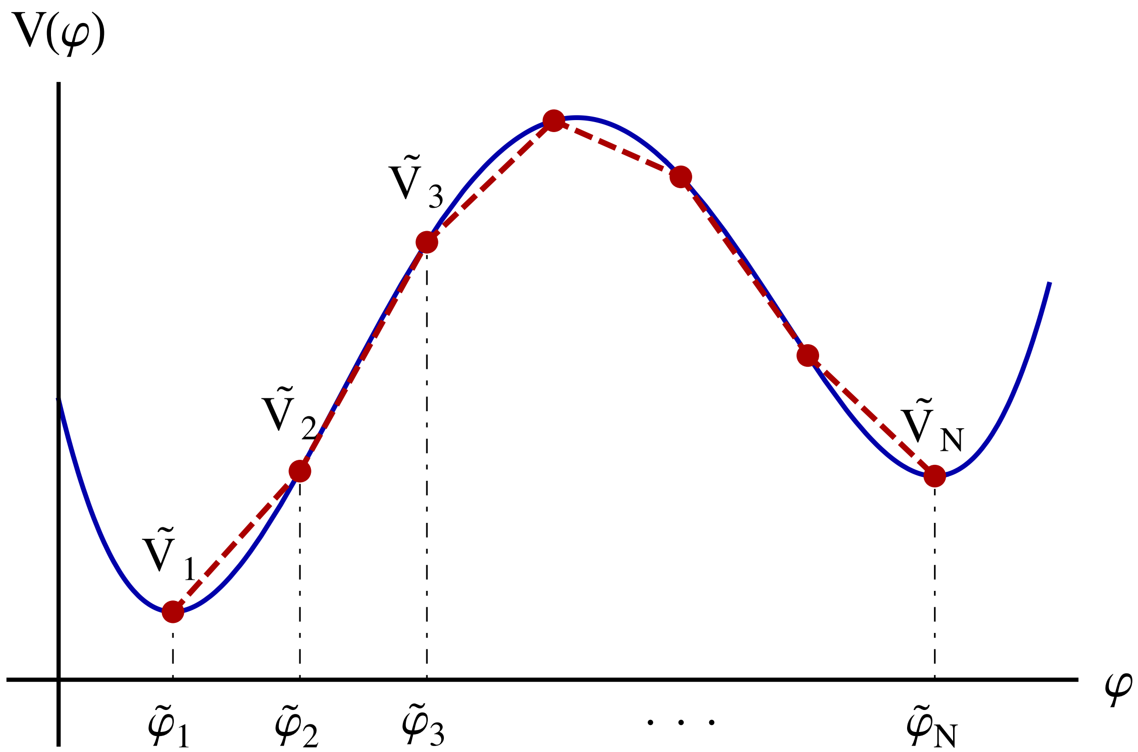


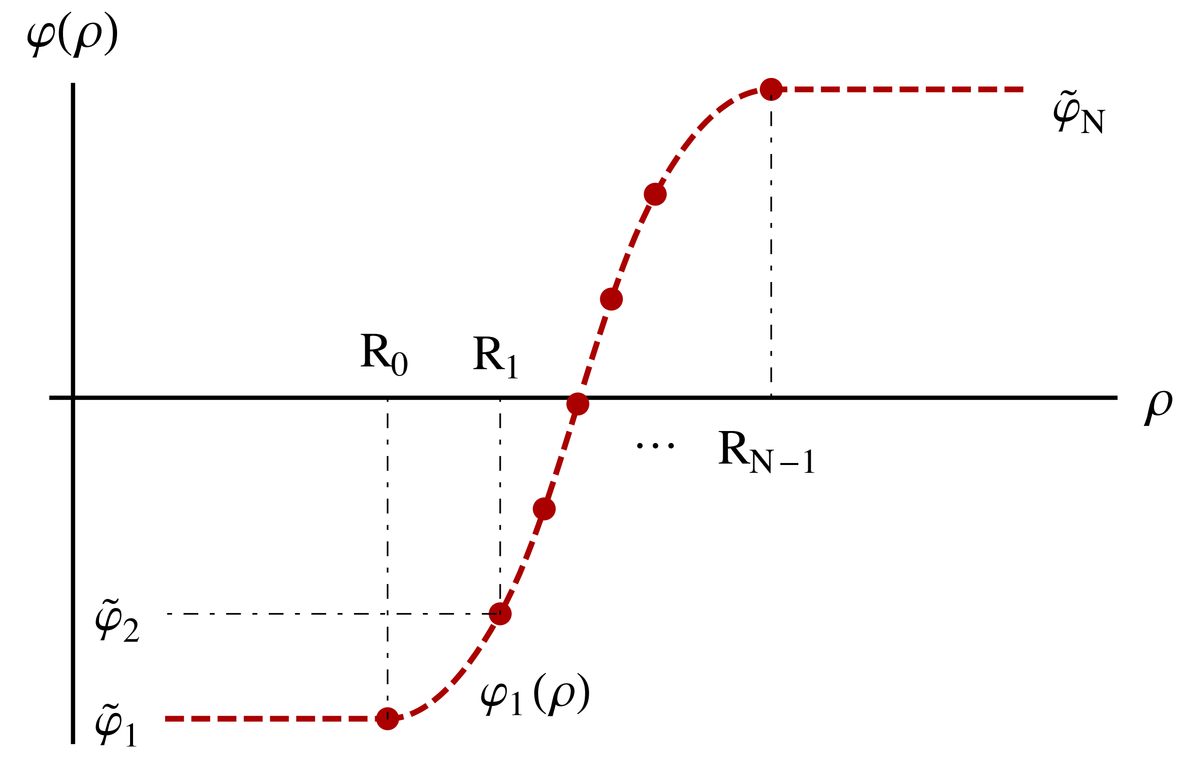
Approximates any V when $N \to \infty$, controlled precision

Geometric insight of segmentation, cover non-trivial features/unstable *V*s

Semi-analytic solution for algebraic manipulation/deformation

Polygonal bounces



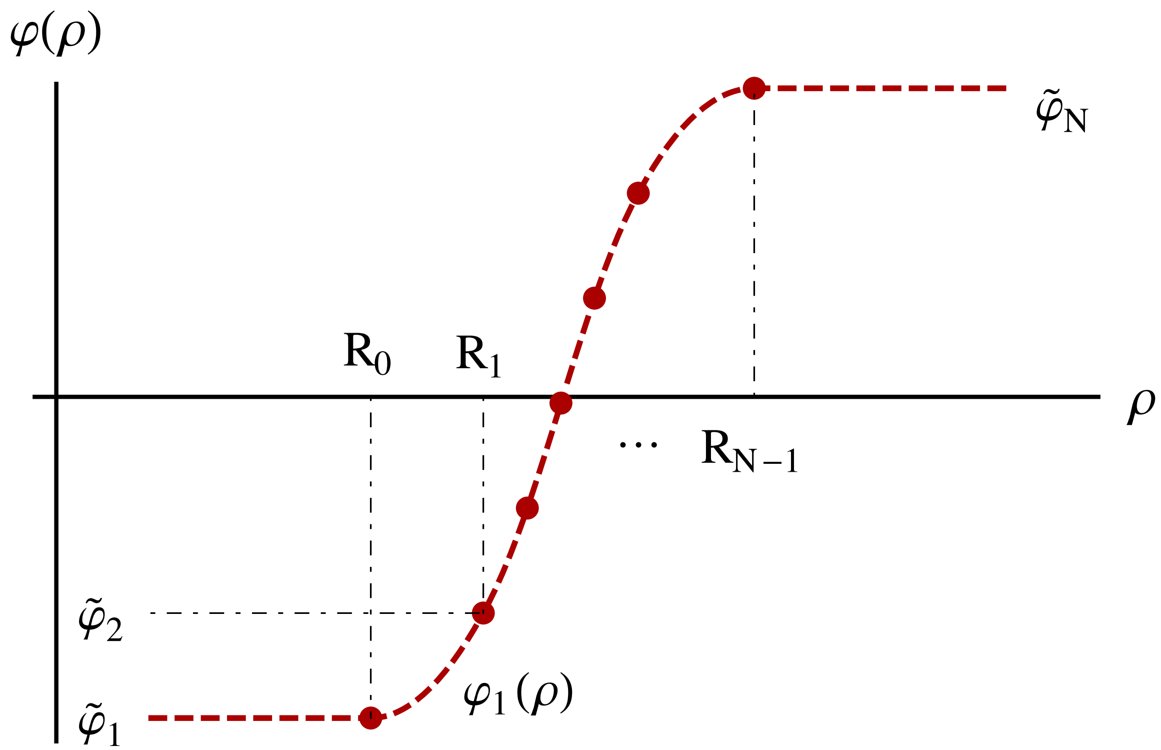


$$V_i(\varphi) = \underbrace{\left(\frac{\tilde{V}_{i+1} - \tilde{V}_i}{\tilde{\varphi}_{i+1} - \tilde{\varphi}_i}\right)}_{8\,a_i}(\varphi - \tilde{\varphi}_i) + \tilde{V}_i - \tilde{V}_N\,, \qquad dV_i = 8\,a_i.$$

No free parameters, one segment three unknowns $v_i,\ b_i,\ R_i$

Generalize case b), solve R_0 or R_i a), retrieve φ_0

Polygonal construction



$$\ddot{\varphi}_i + \frac{D-1}{\rho}\dot{\varphi}_i = dV_i = 8\,a_i$$

$$\varphi_i = v_i + \frac{4}{D}a_i\,\rho^2 + \frac{2}{D-2}\frac{b_i}{\rho^{D-2}}$$

Initial/final conditions remain the same

Matching conditions @ R_i 3 parameters and 3 unknowns/segment

$$\varphi_i(R_1) = \varphi_{i+1}(R_i) = \tilde{\varphi}_{i+1}, \quad \dot{\varphi}_i(R_i) = \dot{\varphi}_{i+1}(R_i)$$

The bounce defined recursively

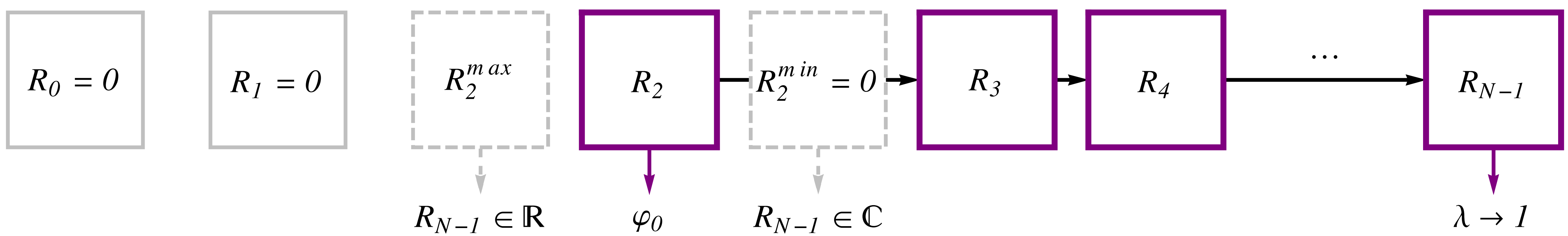
- a) $R_0 = 0$ $$v_n = \varphi_0 - \frac{4}{D-2}\left(a_1 R_0^2 + \sum_{i=1}^{n-1}(a_{i+1} - a_i)\,R_i^2\right)$$
- b) $\varphi_0 = \tilde{\varphi}_1$ $$b_n = \frac{4}{D}\left(a_1 R_0^D + \sum_{i=1}^{n-1}(a_{i+1} - a_i)\,R_i^D\right)$$

Radii computed at each segment from matching the fields $\varphi_n(R_n) = \tilde{\varphi}_{n+1}$

fewnomial

$$R_n^D - \frac{D}{4}\frac{\delta_n}{a_n}R_n^{D-2} + \frac{D}{2(D-2)}\frac{b_n}{a_n} = 0 \qquad \delta_n = \tilde{\varphi}_{n+1} - v_n$$

require real positive roots

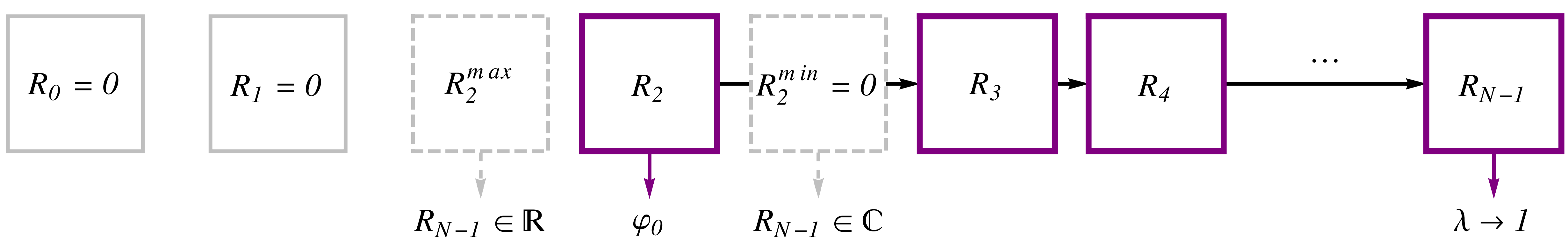


Radii computed at each segment from matching the fields $\varphi_n(R_n) = \tilde{\varphi}_{n+1}$

fewnomial

$$R_n^D - \frac{D}{4}\frac{\delta_n}{a_n} R_n^{D-2} + \frac{D}{2(D-2)}\frac{b_n}{a_n} = 0 \qquad \delta_n = \tilde{\varphi}_{n+1} - v_n$$

require real positive roots



radii solutions

$$D = 3: \quad 2R_n = \frac{1}{\sqrt{a_n}}\left(\frac{\delta_n}{\xi} + \xi\right), \qquad \xi^3 = \sqrt{36 a_n b_n^2 - \delta_n^3} - 6\sqrt{a_n} b_n,$$

simple cubic

$$D = 4: \quad 2R_n^2 = \frac{1}{a_n}\left(\delta_n + \sqrt{\delta_n^2 - 4 a_n b_n}\right)$$

quadratic

$D = 2, 6, 8$ in the paper, other *D*s possible numerically — Amariti '20

Bounce action

Euclidean action
$$S_D = \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \int_0^\infty \rho^{D-1}\, \mathrm{d}\rho \left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right)$$

PB action
$$S_{>2} = \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \left\{ \frac{R_0^D}{D}\left(\tilde{V}_1 - \tilde{V}_N\right) + \sum_{i=1}^{N-1} \left[\rho^2 \left(\frac{32 a_i^2 (D+1)\rho^D}{D^2(D+2)} + \frac{16 a_i b_i}{D(D-2)} - \frac{2b_i^2}{\rho^D (D-2)} \right) + \frac{\rho^D}{D}\left(8 a_i \left(v_i - \tilde{\varphi}_i\right) + \tilde{V}_i - \tilde{V}_N\right) \right]_{R_{i-1}}^{R_i} \right\}$$

Total
$$S = \mathcal{T} + \mathcal{V}$$

$$\mathcal{T} \propto \int_0^\infty \rho^{D-1} \mathrm{d}\rho\, \dot{\varphi}^2, \qquad \mathcal{V} \propto \int_0^\infty \rho^{D-1} \mathrm{d}\rho\, \mathrm{V}(\varphi)$$

kinetic　　　　potential

Derrick's theorem

Non-existence of non-trivial static solutions of KG equation, no solitonic scalar 'particles'

Derrick '64

Unstable under re-scaling $\varphi(\rho) \to \varphi(\rho/\lambda)$

$$\lambda \times 0 = 0,$$
$$\lambda \times \infty = \infty$$

$$S_D^{(\lambda)} = \lambda^{D-2}\mathcal{T} + \lambda^D \mathcal{V}$$

change of variables...remain the same

action is extremized at non-scaled values for true solutions

$$\left.\frac{dS_D^{(\lambda)}}{d\lambda}\right|_{\lambda=1} = 0$$

$$(D-2)\mathcal{T} + D\mathcal{V} = 0$$

relation between kinetic and potential

$$\left.\frac{d^2 S_D^{(\lambda)}}{d\lambda^2}\right|_{\lambda=1} < 0$$

Caveat for PB $R \to \lambda R$

Works for $N \gg 1$

Benchmarks

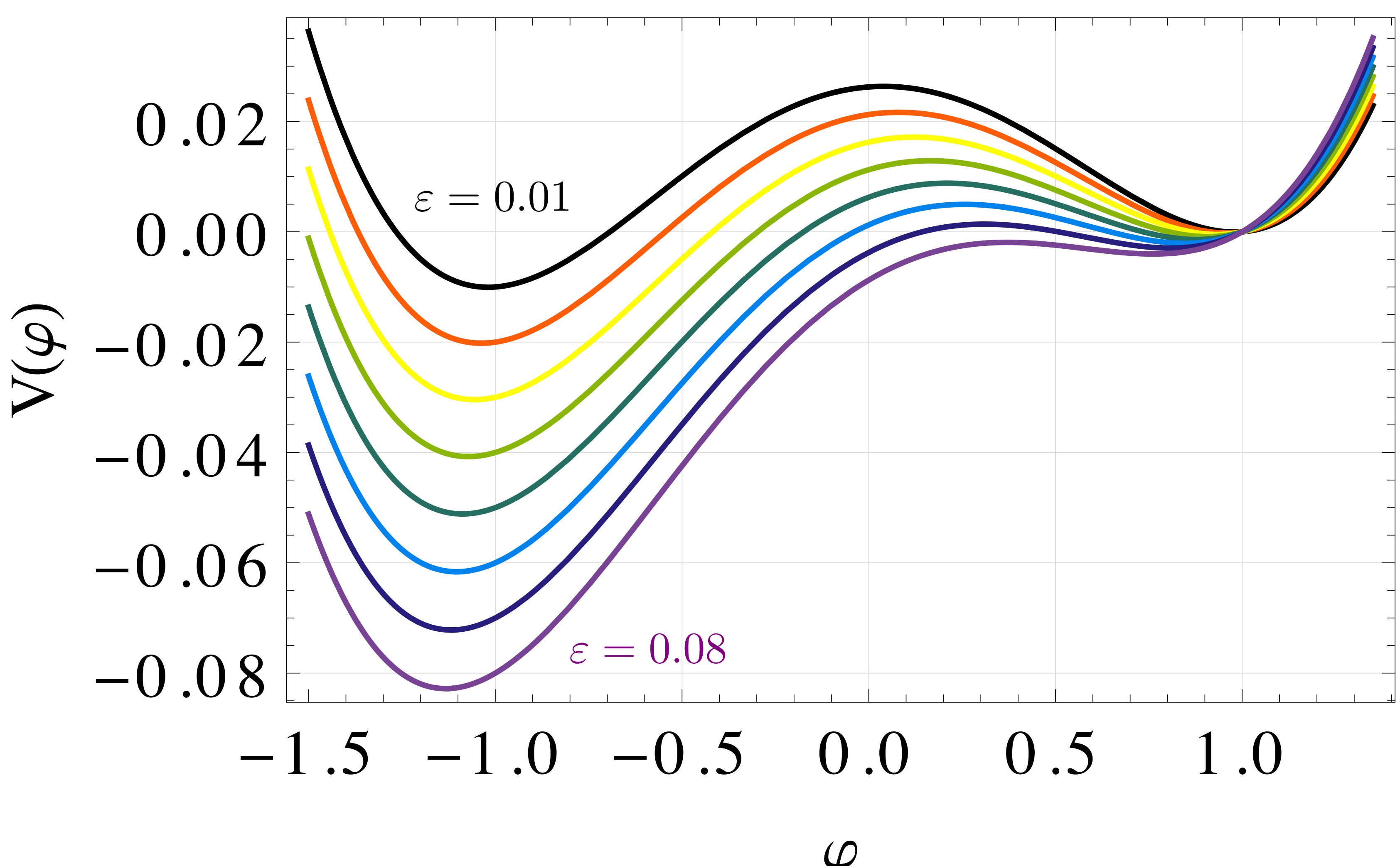
Linearly off-set quartic potential — Coleman ’77

$$V(\varphi) = \frac{\lambda}{8}\left(\varphi^2 - v^2\right)^2 + \varepsilon\left(\frac{\varphi - v}{2v}\right)$$

Benchmark for testing $\lambda = 0.25, \quad v = 1$ — rescaling — Sarid ’98

$\varepsilon \to 0$

thin wall



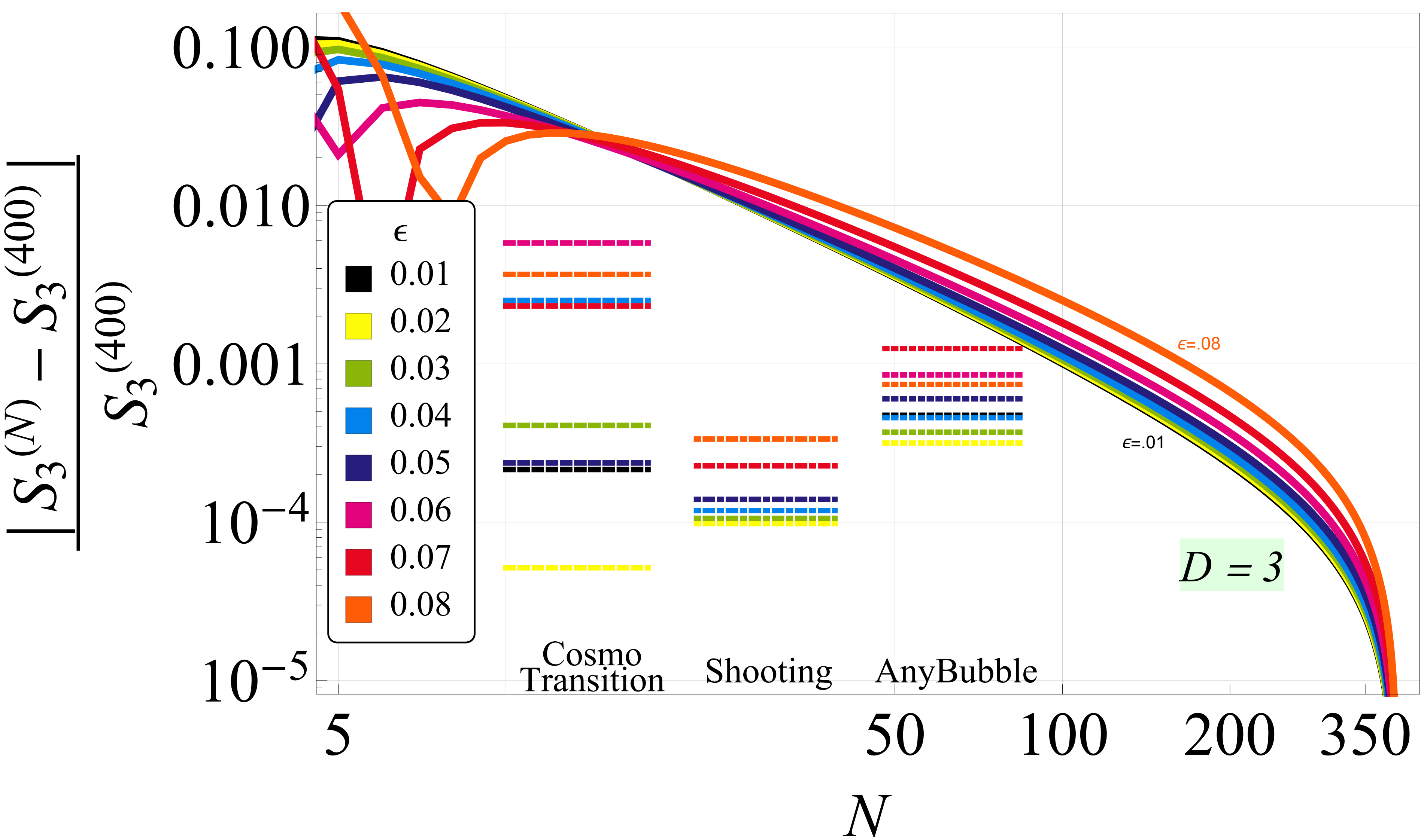
thick wall

$\varepsilon > 0.8$

2nd minimum disappears

Euclidean action, comparisons

- **CosmoTransitions** Runge-Kutta PDE solver, initial value approximations — Wainwright '11
  discontinued
- **AnyBubble** multiple shooting, damping approximations — Masoumi, Olum, Shlaer '16
- **Shooting** Mathematica, precise setting of initial values, issues with 0, infinity



PB within permille after 100 iterations

stable for thin/thick

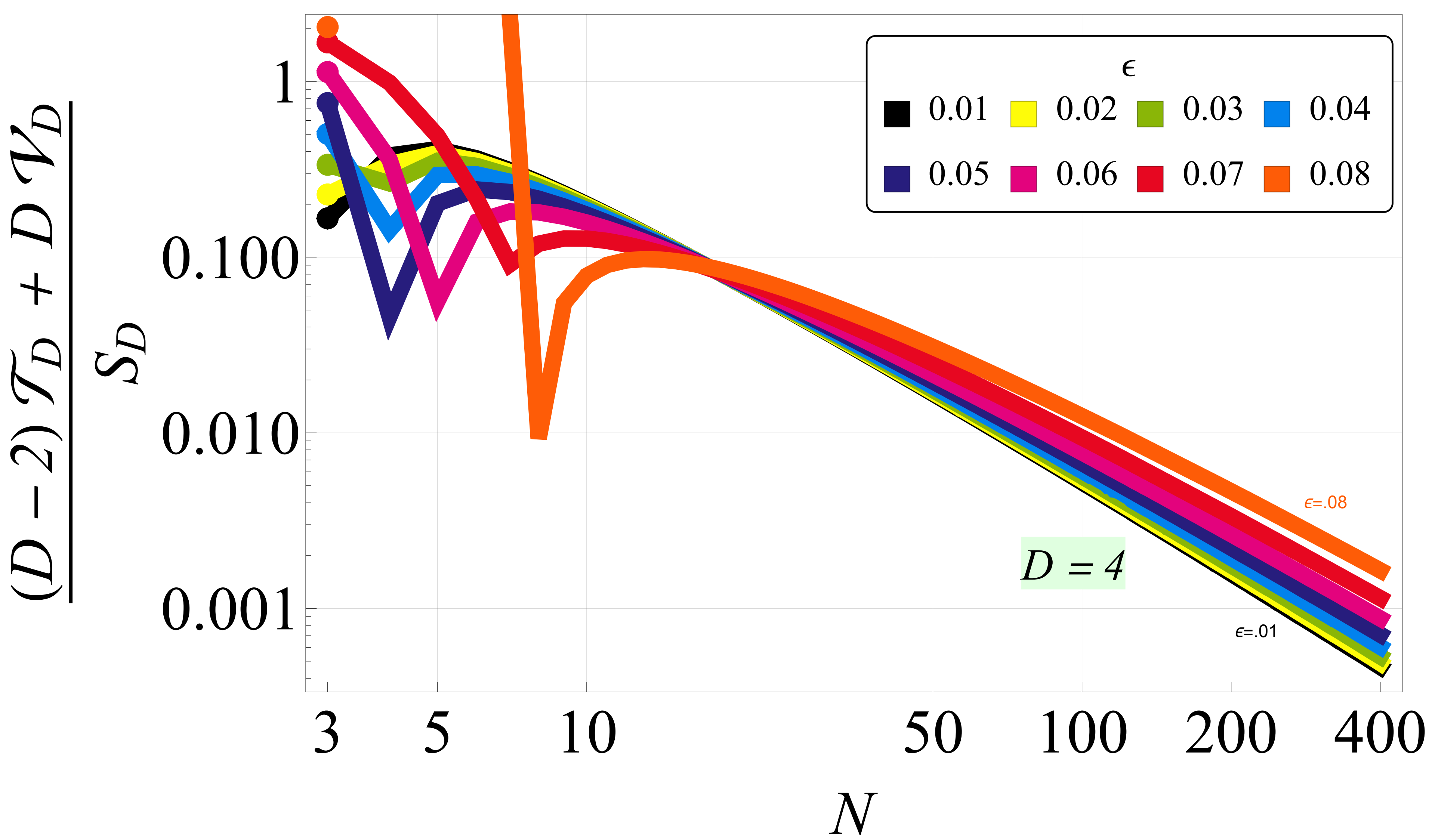
faster convergence for thin wall

Derrick's theorem $$(D-2)\mathcal{T} + D\mathcal{V} \to 0$$

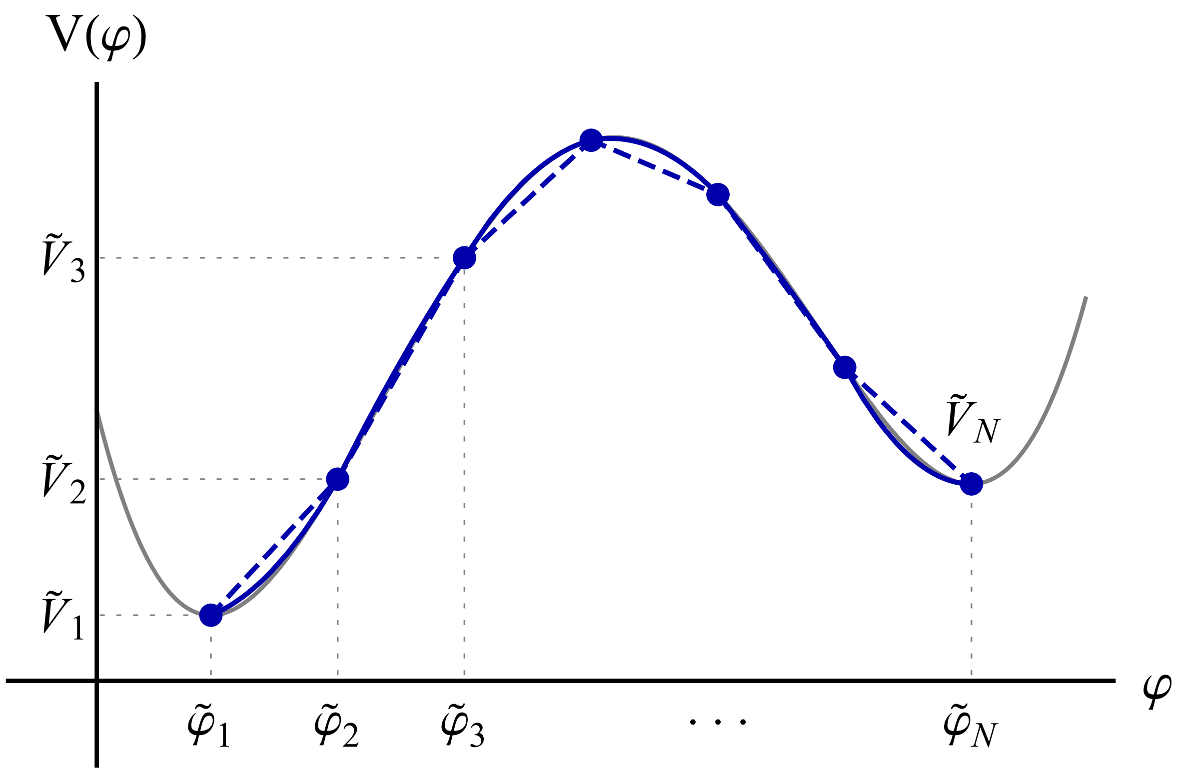
finite part corrections up to $N \simeq 10$

independent measure of goodness of approximation

above relation 'exact' for the PB potential



Higher orders



Expand to higher orders

- improves convergence
- important @ extrema

$$V_i \simeq \tilde{V}_i - \tilde{V}_N + \partial \tilde{V}_i \left(\varphi_i - \tilde{\varphi}_i\right)$$

$$+ \frac{\partial^2 \tilde{V}_i}{2} \left(\varphi_i - \tilde{\varphi}_i\right)^2$$

Perturbative expansion $\quad \varphi = \varphi_{PB} + \xi$

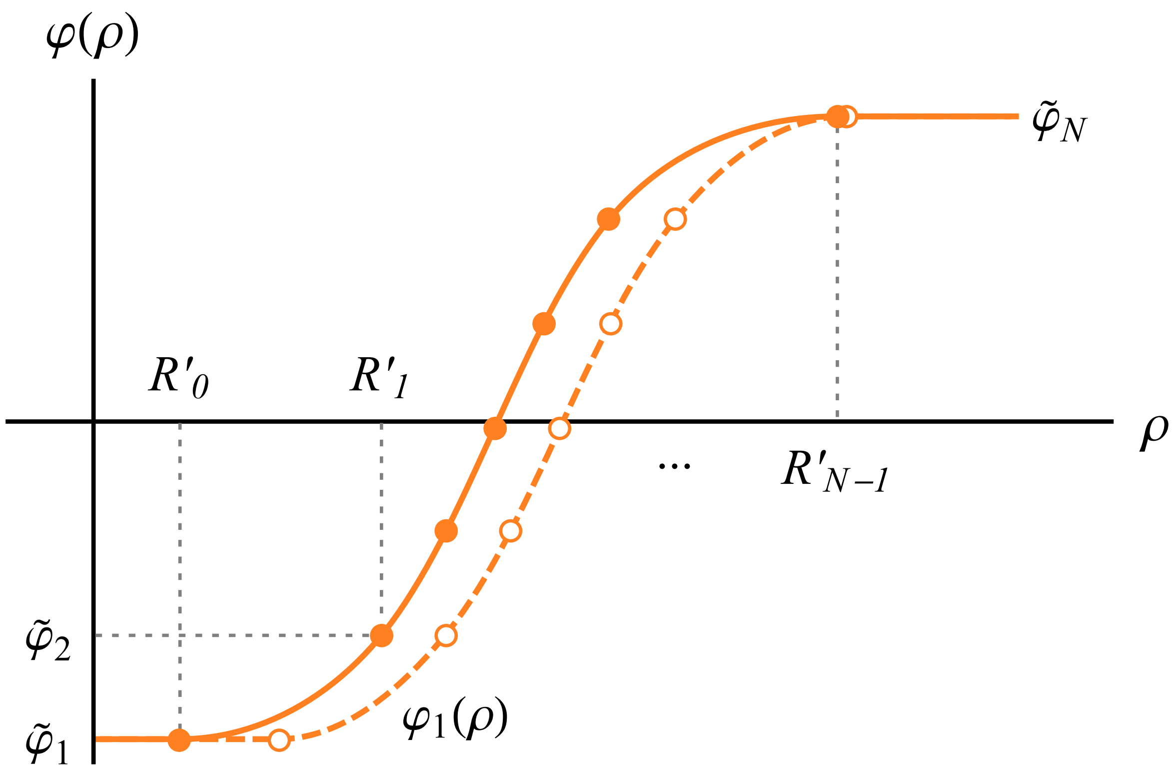
$$\ddot{\varphi} + \frac{D-1}{\rho}\dot{\varphi} = 8\left(a + \alpha\right) + \delta dV\left(\varphi_{PB}(\rho)\right)$$

$$\ddot{\xi} + \frac{D-1}{\rho}\dot{\xi} = 8\alpha + \delta dV(\rho)$$

$$\mathcal{I}_s^{D=4} = \partial^2 \tilde{V}_s \left(\frac{v_s - \tilde{\varphi}_s}{8}\rho^2 + \frac{b_s}{2}\ln\rho + \frac{a_s}{24}\rho^4 \right)$$

$$\xi = \nu + \frac{2}{D-2}\frac{\beta}{\rho^{D-2}} + \frac{4}{D}\alpha\rho^2 + \mathcal{I}(\rho)$$

Higher orders



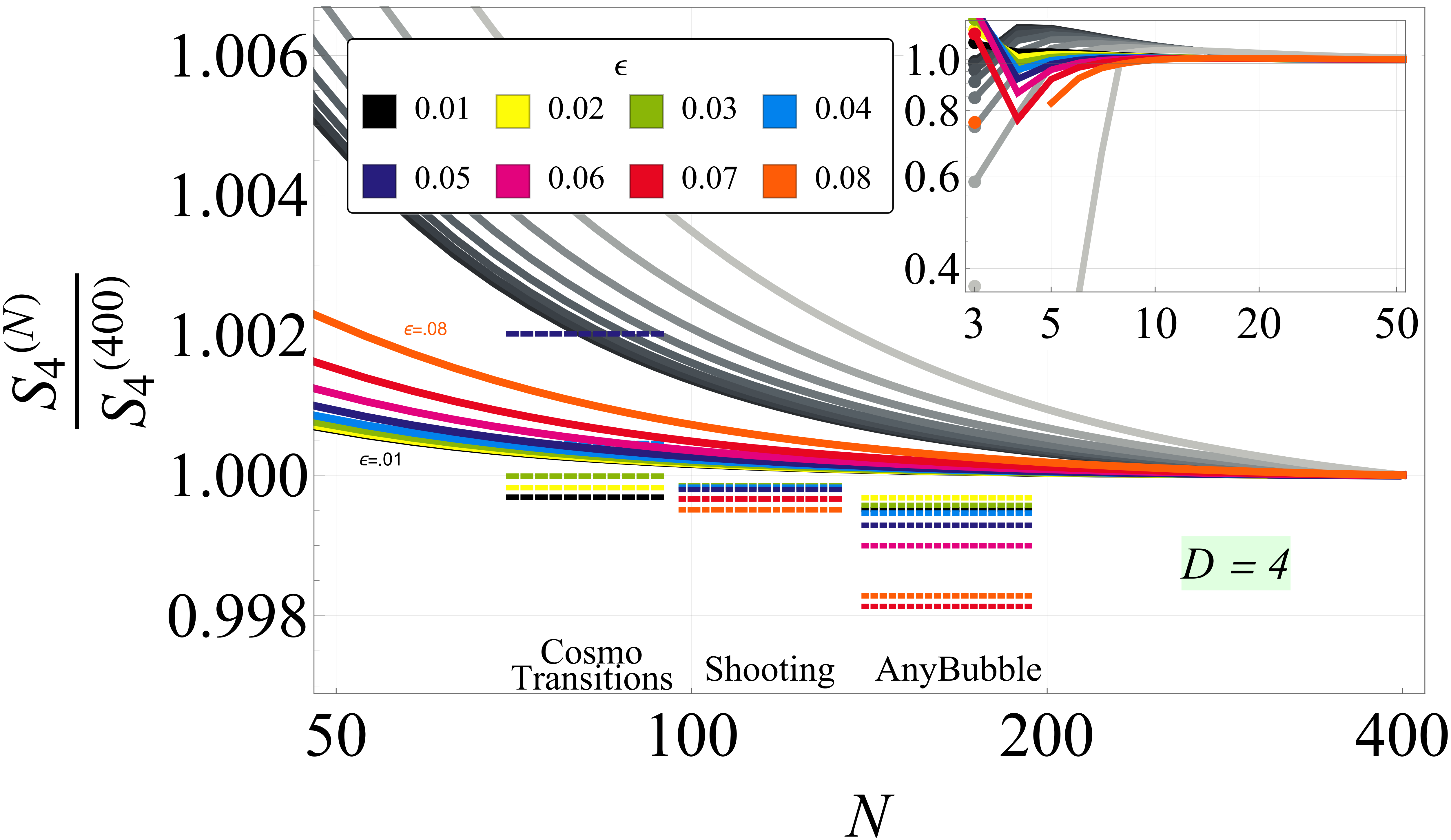
Match at perturbed radii

$$R_s \to R_s\,(1 + r_s)\,, \qquad r_s \ll 1$$

Rederive the matching conditions

A single linear equation = very fast

$$r_s = \frac{\beta_s + \frac{D-2}{2}\left(\nu_s + \mathcal{I}_s + \frac{4}{D}\alpha_s R_s^2\right) R_s^{D-2}}{(D-2)\left(b_s - \frac{4}{D} a_s R_s^D\right)}$$



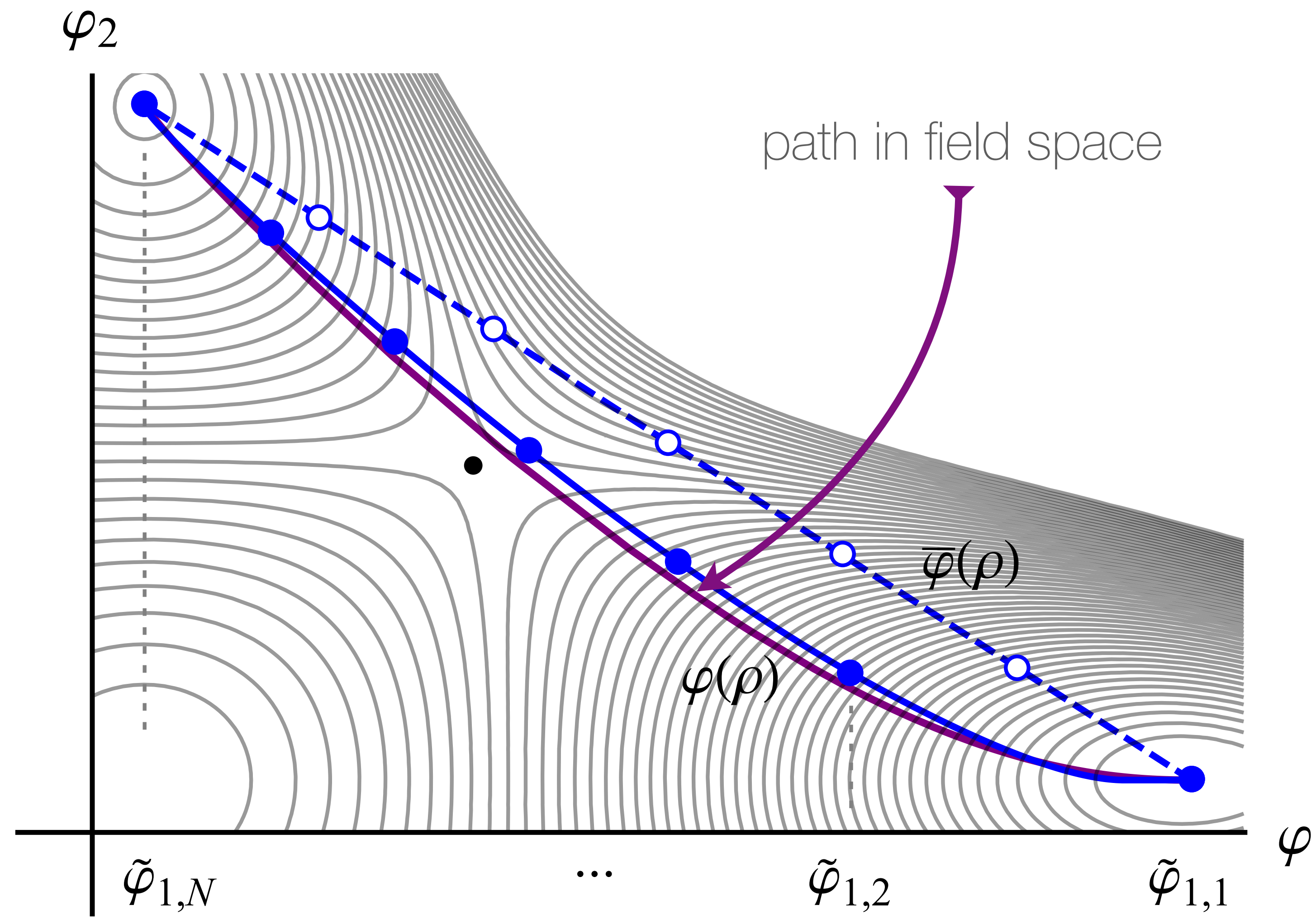
ϵ
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
1.006
1.004
1.002
1.000
0.998
ϵ=.08
ϵ=.01
$\frac{S_4^{(N)}}{S_4^{(400)}}$
Cosmo Transitions
Shooting
AnyBubble
D = 4
50
100
200
400
N
1.0
0.8
0.6
0.4
3
5
10
20
50

Multi-fields

$$\ddot{\varphi}_i + \frac{D-1}{\rho}\dot{\varphi}_i = \frac{dV}{d\varphi_i} \qquad \varphi_i(0) = \varphi_{0i}$$

Shooting method impractical

- highly non-linear
- multi-dimensional field space



Multi-fields

$$\ddot{\varphi}_i + \frac{D-1}{\rho}\dot{\varphi}_i = \frac{dV}{d\varphi_i} \qquad \varphi_i(0) = \varphi_{0i}$$

- CosmoTransitions — Wainwright '11

  bounce and path deformation separate, oscillations, Runge-Kutta PDE solver

- AnyBubble — Masoumi, Olum, Shlaer '16

  multiple shooting, damping approximations

- Other recent approaches

  tunnelling potential

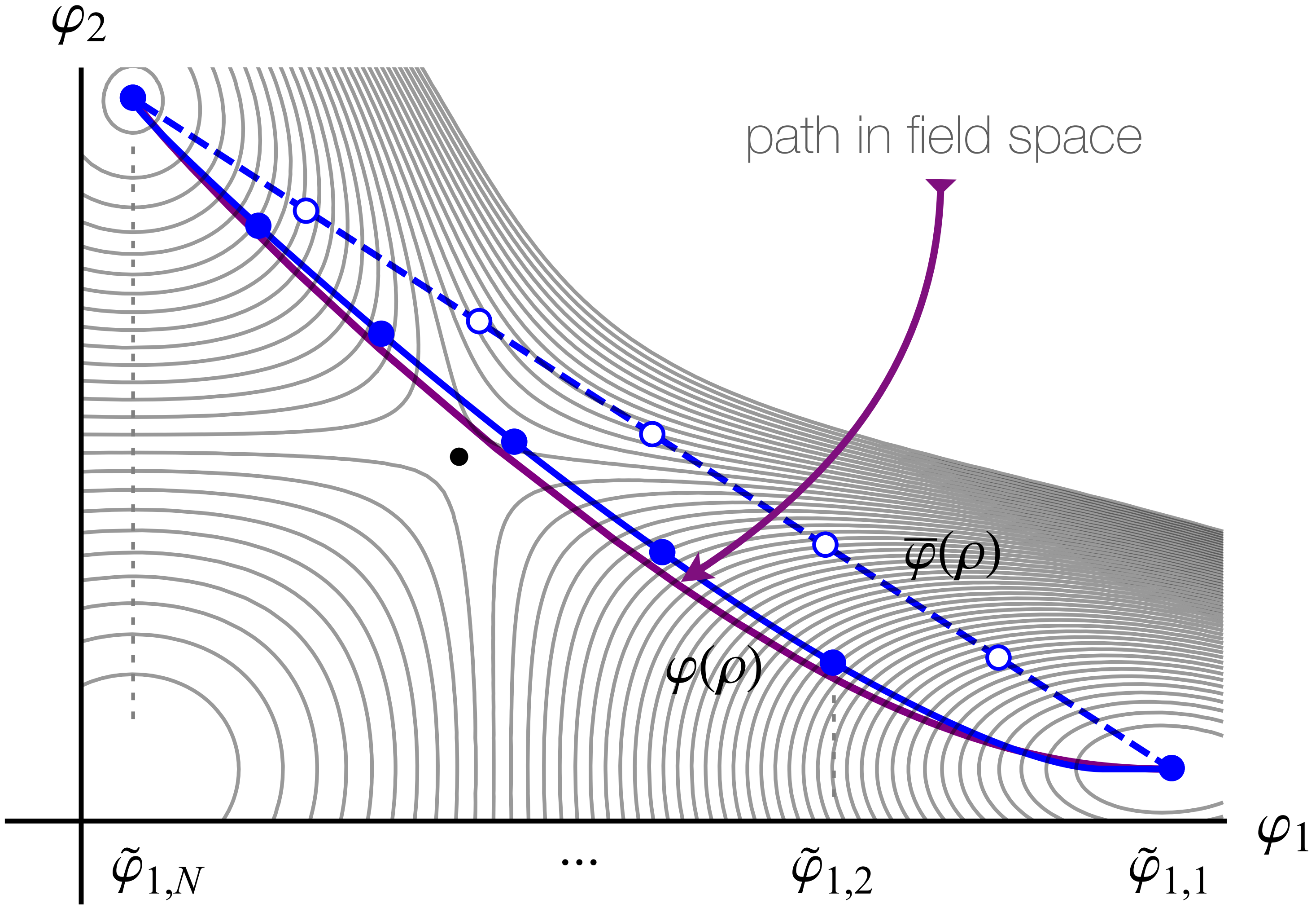
  Espinosa, Konstandin '18

  machine learning

  Piscopo, Spannowsky, Waite '19

  gradient flow

  Sato '19

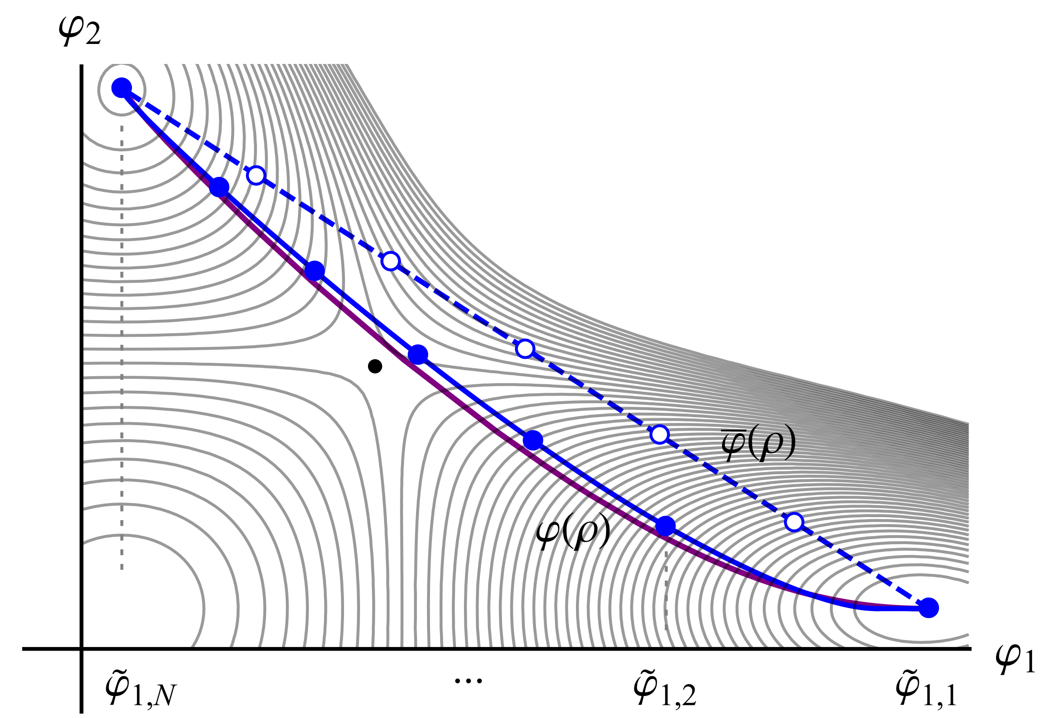


Polygonal approach with many fields

- **Initial ansatz** straight line, via saddle, custom segmentation
- **Initial solution** longitudinal single field PB

$\tilde{\varphi}_{is}$

$\bar{\varphi}_{is}(\rho)$



Crucial idea #1

- perturbation up to linear term in V, keeps the PB

$$\underbrace{\ddot{\bar{\varphi}}_{is} + \frac{D-1}{\rho}\dot{\bar{\varphi}}_{is}}_{8\bar{a}_{is}} + \underbrace{\ddot{\zeta}_{is} + \frac{D-1}{\rho}\dot{\zeta}_{is}}_{8a_{is}} = \frac{dV}{d\varphi_i}(\bar{\varphi}+\zeta)$$

$$\zeta_{is} = v_{is} + \frac{2}{D-2}\frac{b_{is}}{\rho^{D-2}} + \frac{4}{D}a_{is}\rho^2$$

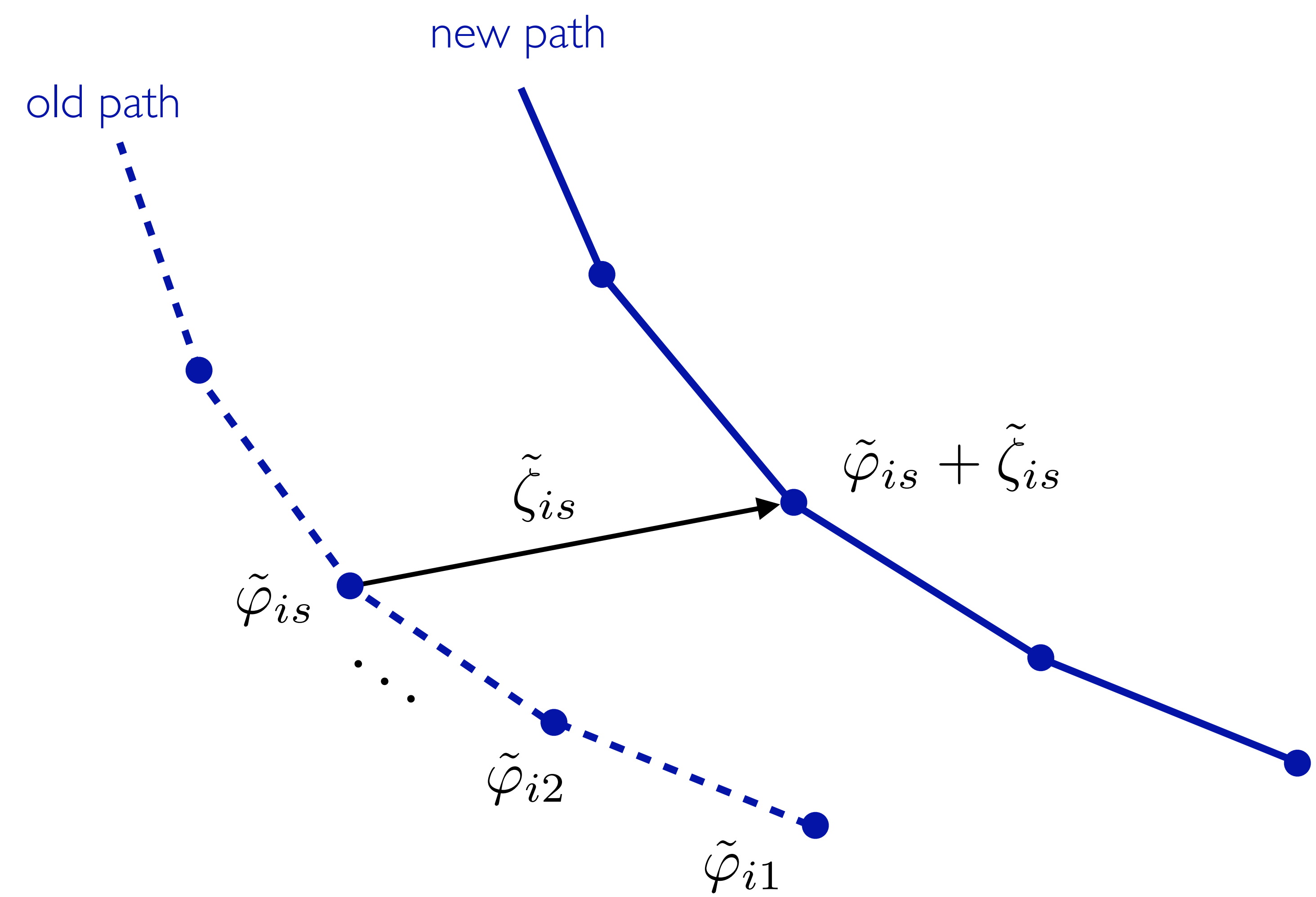
$$\underbrace{\ddot{\bar{\varphi}}_{is} + \frac{D-1}{\rho}\dot{\bar{\varphi}}_{is}}_{8\bar{a}_{is}} + \underbrace{\ddot{\zeta}_{is} + \frac{D-1}{\rho}\dot{\zeta}_{is}}_{8a_{is}} = \frac{dV}{d\varphi_i}\left(\bar{\varphi} + \zeta\right)$$

Crucial idea #2

$$8a_{is} \simeq \frac{dV}{d\varphi_i}\left(\tilde{\varphi}_{is} + \tilde{\zeta}_{is}\right) - 8\bar{a}_{is}$$

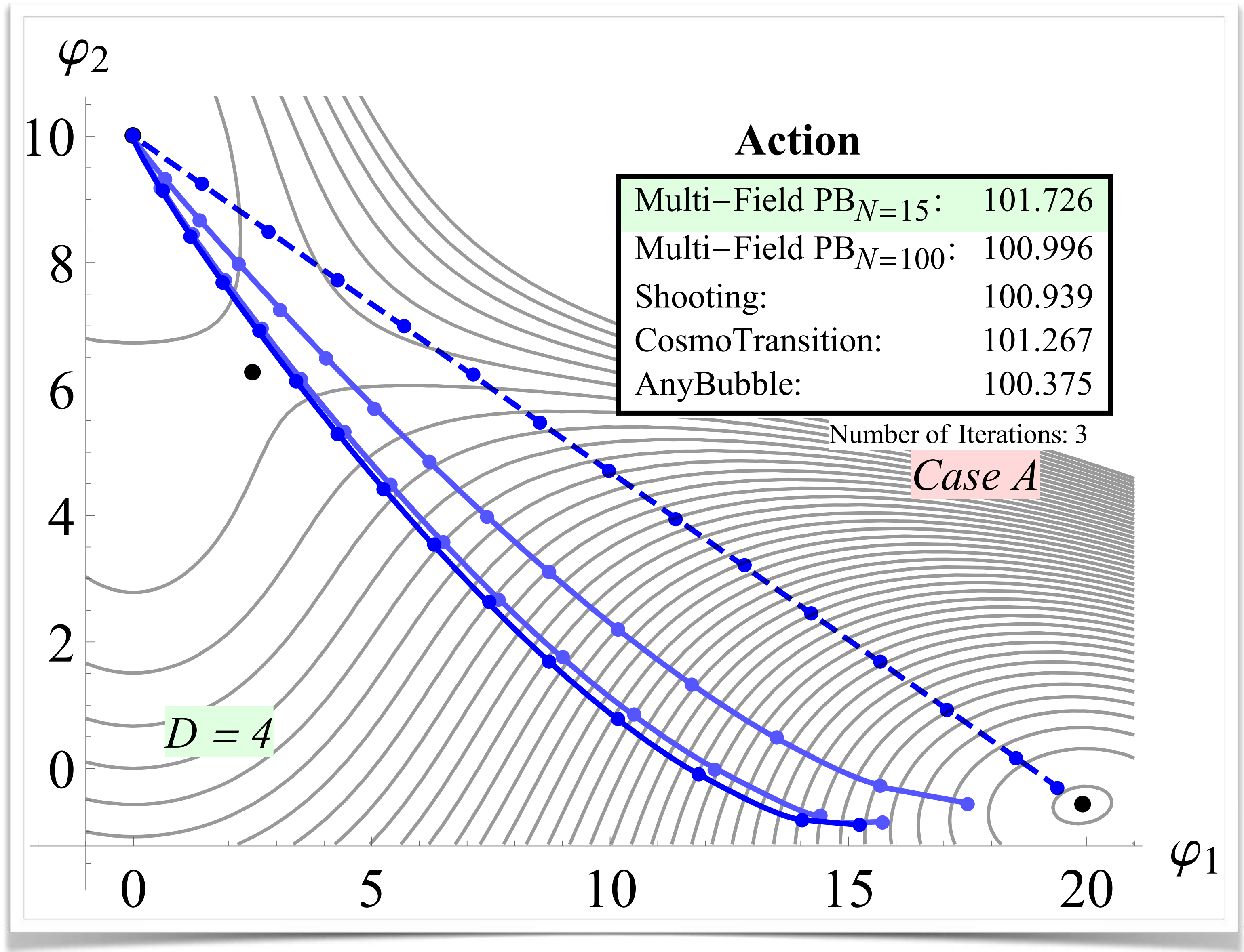
$$\frac{dV}{d\varphi_i} \simeq \frac{1}{2}\left(d_i\tilde{V}_s + d_i\tilde{V}_{s+1} + d^2_{ij}\tilde{V}_s\tilde{\zeta}_{js} + d^2_{ij}\tilde{V}_{s+1}\tilde{\zeta}_{js+1}\right)$$

- simultaneous solution for the bounce and path deformation
- linear system for r_{i0} (as in the single field expansion) and $\tilde{\zeta}_{is}$
- iterate until $\tilde{\zeta}_{is} < \varepsilon_{\Delta\varphi}$



$$V\left(\varphi_i\right)=\sum_{i=1}^{2}\left(-\mu_i^2\varphi_i^2+\lambda_i^2\varphi_i^4\right)+\lambda_{12}\varphi_1^2\varphi_2^2+\tilde{\mu}^3\varphi_2$$

- no oscillations
- converges in a few iterations
- works for thin wall
- works for *D=3* and *4*
- tested for up to 20 fields

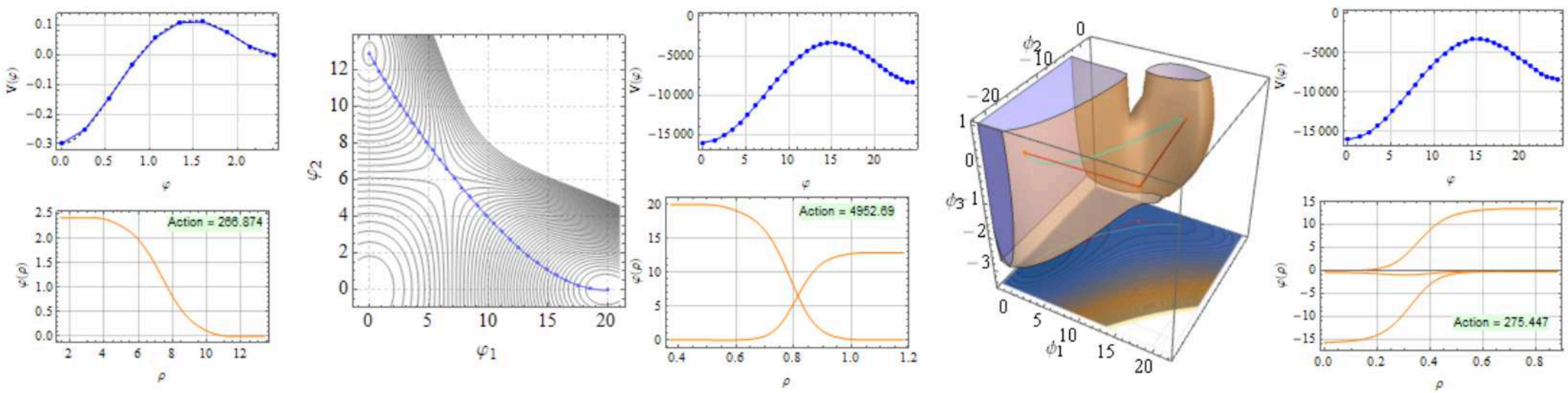


FindBounce

FindBounce https://github.com/vguada/FindBounce/releases

FindBounce is a Mathematica package that computes the bounce configuration needed to compute the false vacuum decay rate with multiple scalar fields.

We kindly ask the users of the package to cite the two papers that describe the working of the *FindBounce* package: the paper with the original proposal by Guada, Maiezza and Nemevšek (2019) and the software release manual by Guada, Nemevšek and Pintar (2020).



Installation

To use the *FindBounce* package you need Mathematica version 10.0 or later. The package is released in the `.paclet` file format that contains the code, documentation and other necessary resources. Download the latest `.paclet` file from the repository "releases" page to your computer and install it by evaluating the following command in the Mathematica:

```
(* Path to .paclet file downloaded from repository "releases" page. *)
PacletInstall["full/path/to/FindBounce-X.Y.Z.paclet"]
```

Load the package as usual

```
In[1]:= Needs["FindBounce`"]
```

Define a metastable potential

```
In[2]:= V[x_]:= 0.5 x^2 + 0.5 x^3 + 0.12 x^4;

In[3]:= extrema = x/.Sort@Solve[D[V[x],x]==0];
```

Compute the bounce - obtain bf = the bounce function

```
In[4]:= bf = FindBounce[V[x],x,{extrema[[1]],extrema[[3]]}]
```

```
Out[4]= BounceFunction[ ⊞ Action: 73500. Dimension: 4 ]
```

```
In[5]:= bf["Action"]

Out[5]= 73496.

In[6]:= bf["Dimension"]

Out[6]= 4
```

Retrieve the bounce properties

Run on single points `FindBounce[{{x1,V1},{x2,V2},...}]`

And multifields `FindBounce[V[x,y,...],{x,y,...},{m1,m2}]`

Custom options control the input

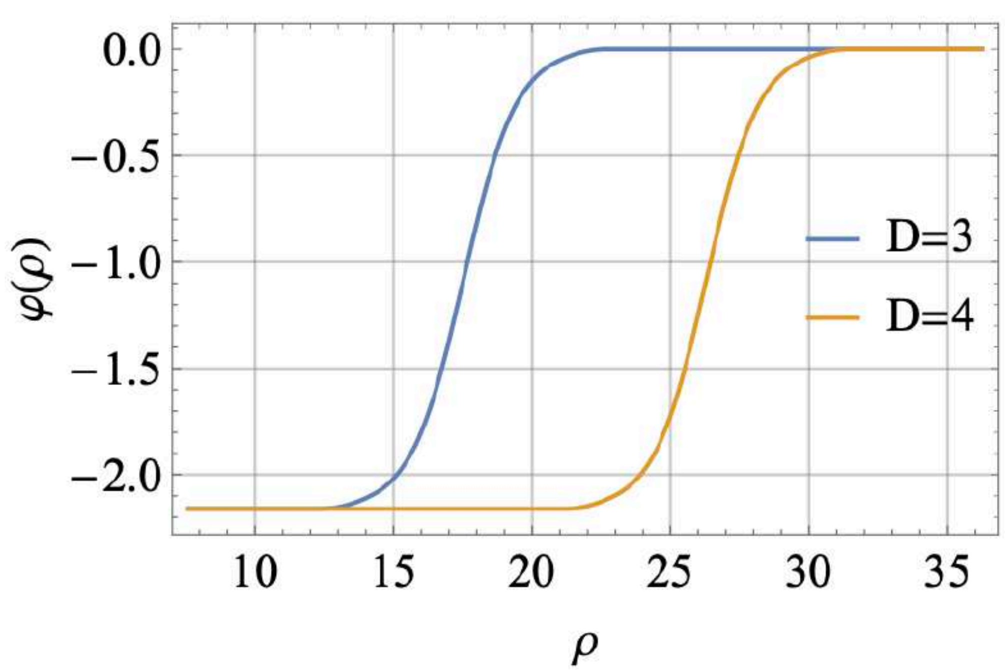
`"Dimension"`	sets the Eucludiean spacetime dimension, 3 or 4
`"FieldPoints"`	number of field points defines the segmentation
`"Gradient"`	one can pre-calculate the gradient, or make it numerical
`"Hessian"`	similar to the gradient, needed for multi-fields
`"MaxPath"`	limits the number of path iterations, typically small
`"MidFieldPoint"`	one can define a starting fixed point (e.g. the saddle)
`"PathTolerance"` & `"ActionTolerance"`	set a goal for the precision of the path variation and the Euclidean action

Output is a bundled container that can be easily accessed

`"Action"`	sets the Euclidean spacetime dimension, 3 or 4
`"Bounce"`	number of field points defines the segmentation
`"Coefficients"`	one can pre-calculate the gradient, or make it numerical
`"Path"`	similar to the gradient, needed for multi-fields
`"Radii"`	limits the number of path iterations, typically small

```
In[9]:= BouncePlot[{bf3,bf}, PlotLegends-> Placed[{"D=3","D=4"}, {Right,Center}]]
```

Out[9]=



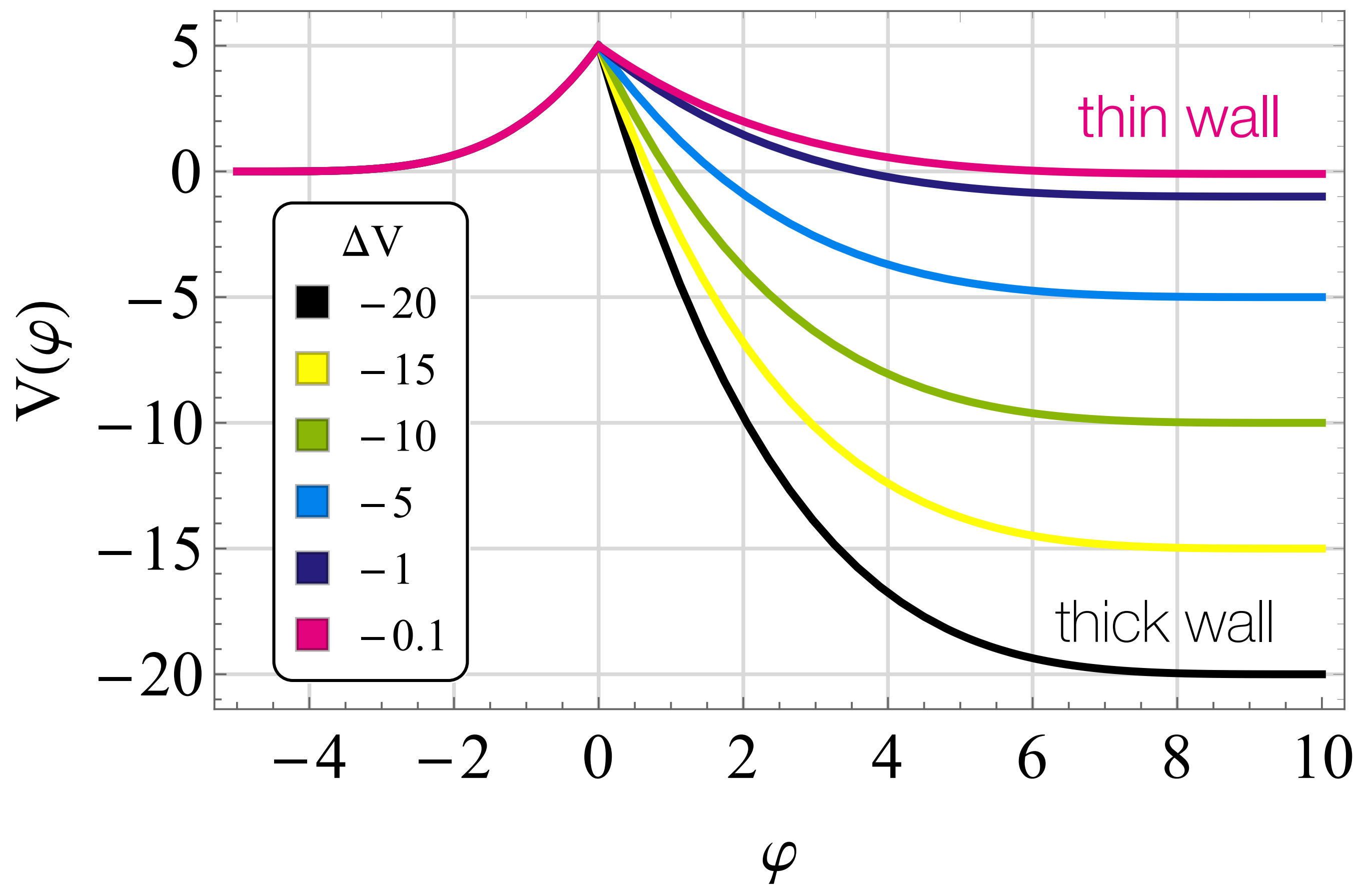
custom function for convenient plotting

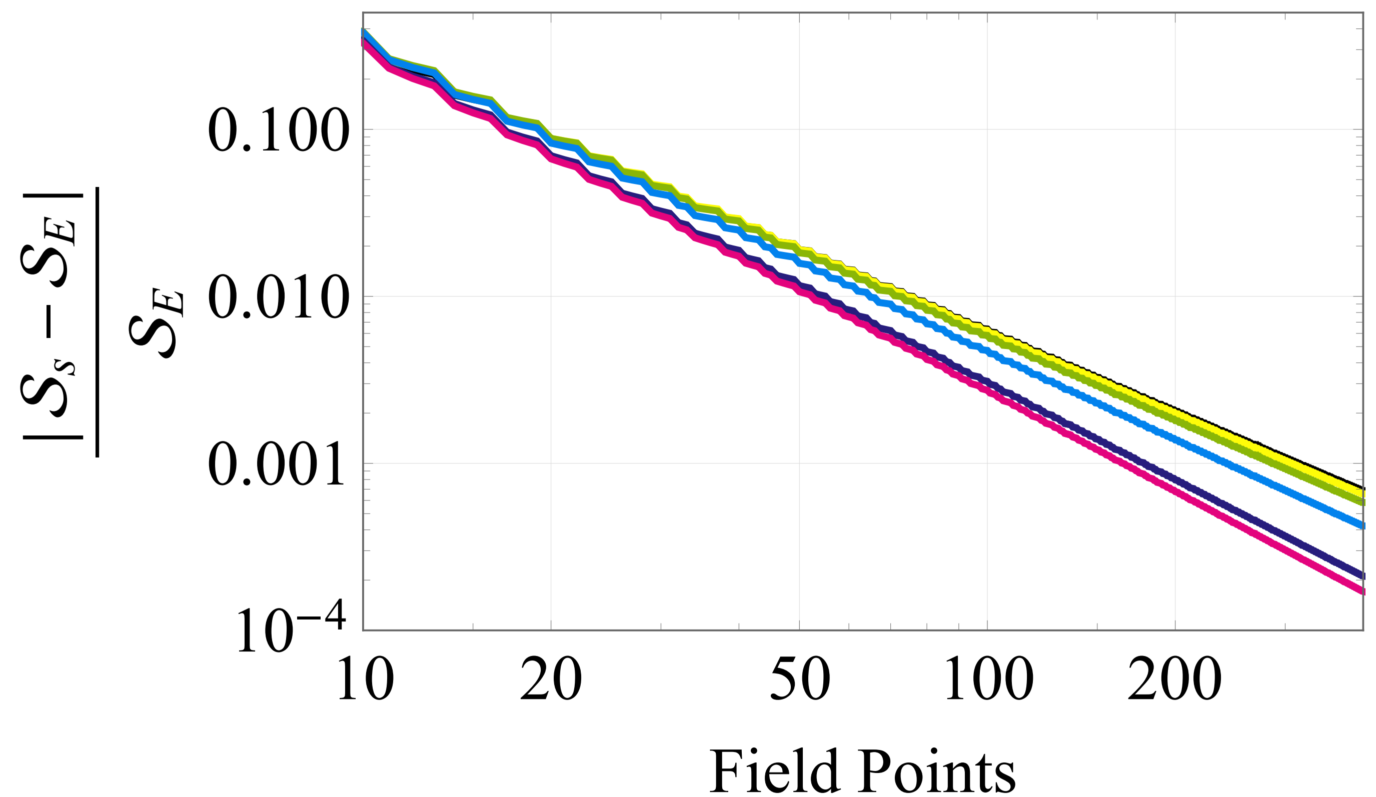
Bi-quartic

Other exact *N=3* potentials, quartic-linear, quartic-quartic

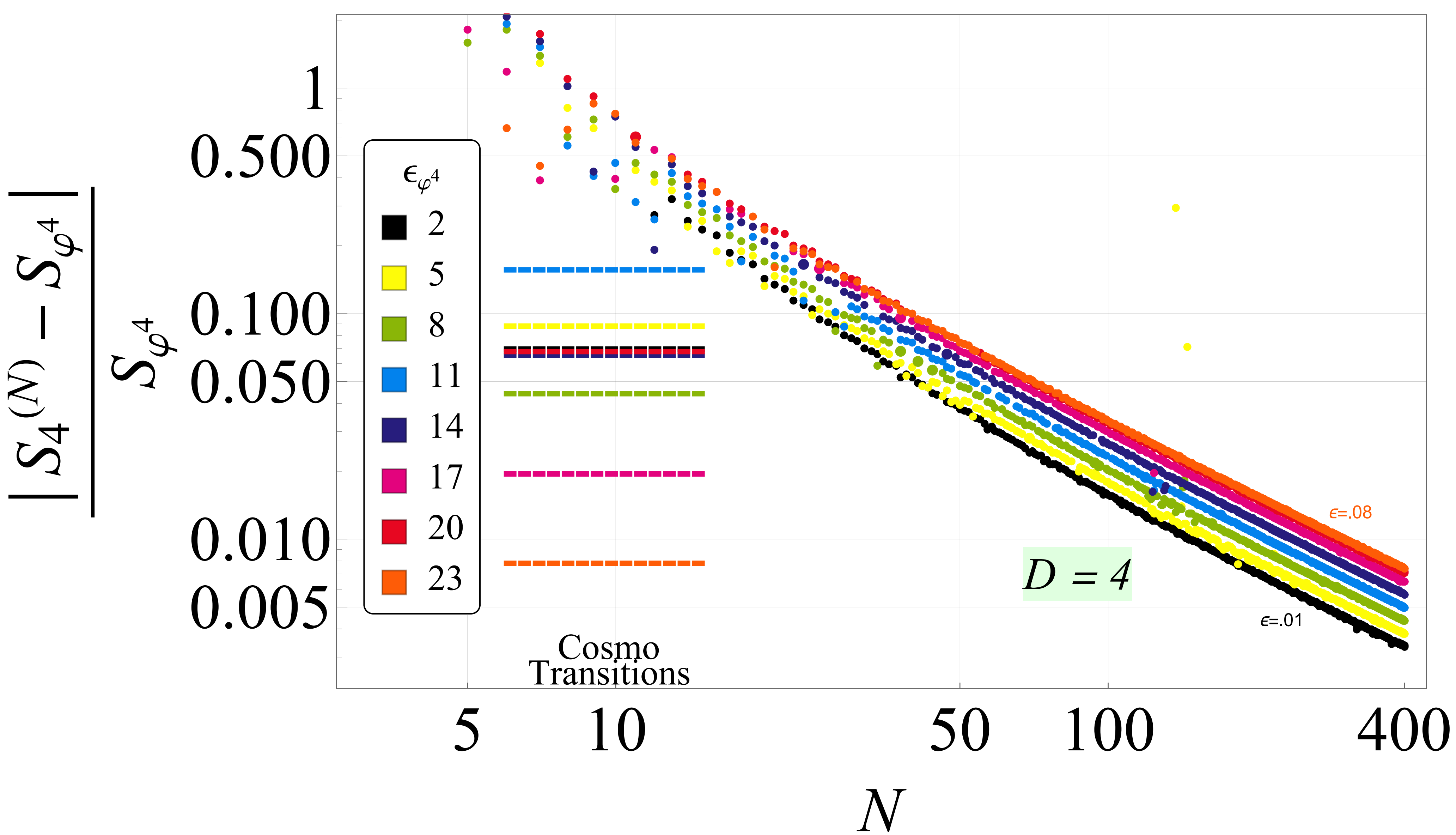
Dutta, Hector, Vaudrevange, Westphal '11

known exact solution, 'fair' comparison and test for the PB method

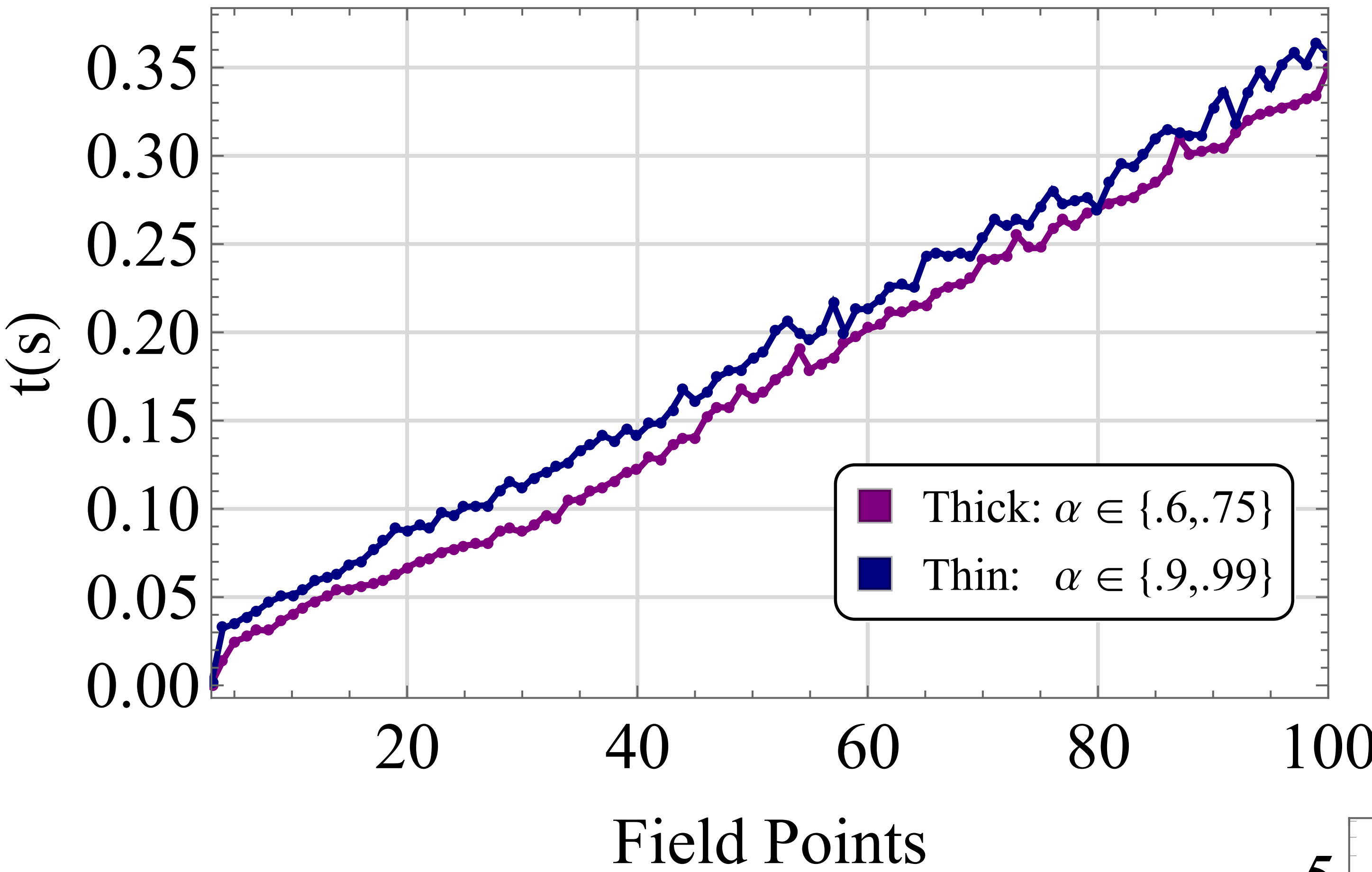




- **CosmoTransitions** — fails with the action, possible to repair by hand, precise from 20% to 0.5%
- **AnyBubble** — fails to compute
- **Polygonal bounce** — works smoothly with a bi-homogeneous segmentation



Time demand



Scales linearly by construction

Works in thin and thick regimes

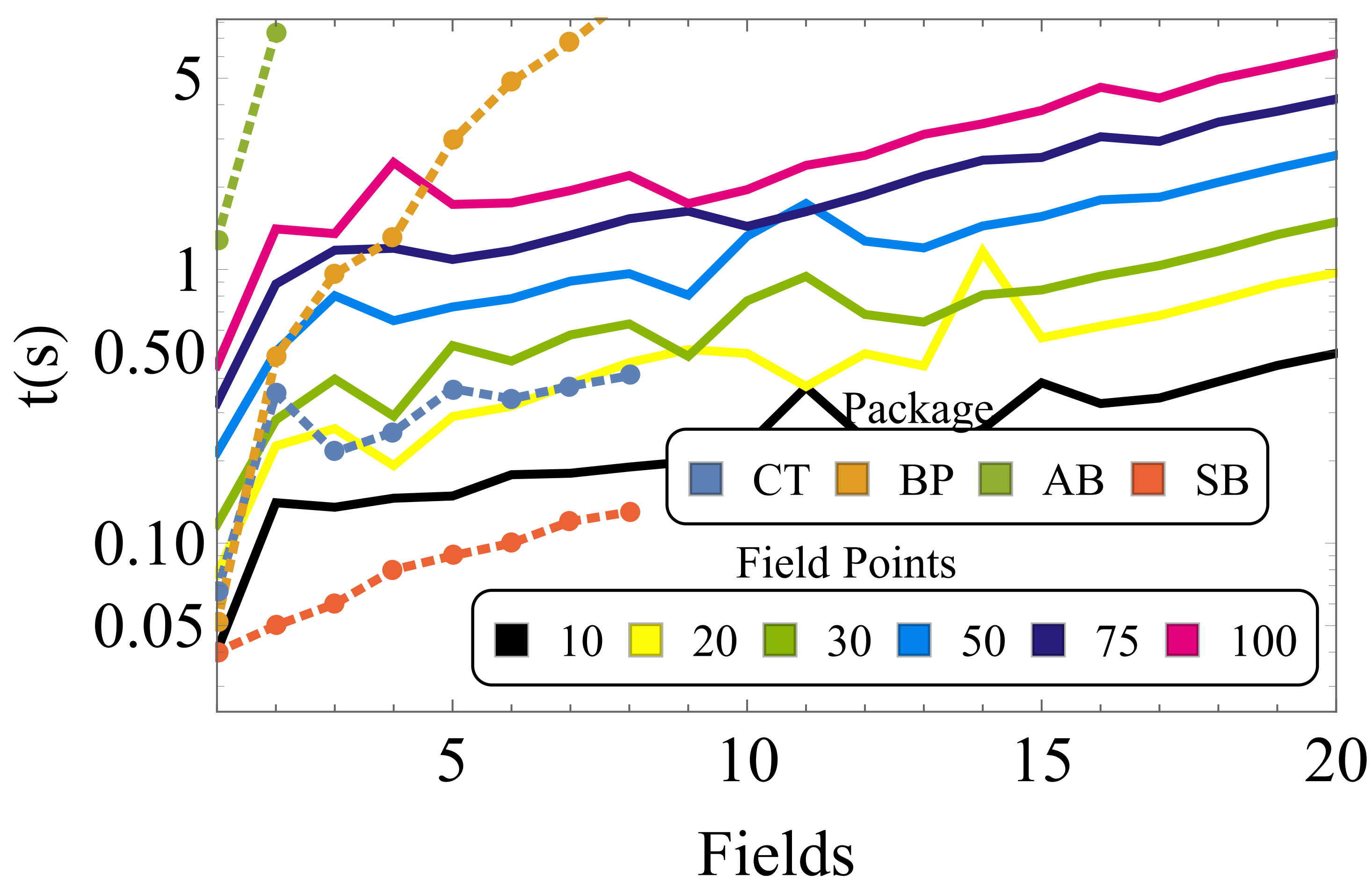
Tested up to 20 fields

CT - CosmoTransitions Wainwright '11

BP - BubbleProfiler Masoumi et al. '16

AB - AnyBubble Athron et al. '19

SB - SimpleBounce Sato '20



Quantum Fluctuations

$$\frac{\Gamma}{\mathcal{V}} = \int \mathcal{D}\varphi \, e^{-S[\varphi]}$$

path integral(s) around the bounce $\varphi = \overline{\varphi} + \psi$

$$S[\varphi] \simeq S[\overline{\varphi}] + \frac{\delta S}{\delta \varphi}\bigg|_{\overline{\varphi}} \psi + \frac{1}{2} \psi \frac{\delta^2 S}{\delta \varphi^2}\bigg|_{\overline{\varphi}} \psi + \dots$$

Fluctuations are the prefactor

Gives the dimensional overall energy scale (from zero removal)

Formally a one loop contribution, infinite and has to be regularized

Needed when large logs are present to cancel the running of the bounce action

Path integrals

PI from QM $$\frac{\Gamma}{\mathcal{V}} = \int \mathcal{D}\varphi \, e^{-S[\varphi]}$$ Feynman, Hibbs ’65

QFT seminal work Callan, Coleman ’77

$$\mathcal{O} = \frac{\delta^2 S}{\delta \varphi^2} \qquad \mathcal{O}\psi_n = \lambda_n \psi_n \qquad \int \mathcal{D}\varphi \to \prod_n \frac{1}{\sqrt{\lambda_n}}$$

Eigenvalues describe the bubble deformation, zeroes for any symmetry, single negative

$$\frac{\Gamma}{\mathcal{V}} = \frac{\mathrm{Im} \int \mathcal{D}\varphi \, e^{-S[\overline{\varphi}]}}{\int \mathcal{D}\varphi \, e^{-S[\varphi_{\mathrm{FV}}]}} = \left(\frac{\mathcal{S}_0}{2\pi} \right)^2 e^{-\mathcal{S}_0} \, \mathrm{Im} \sqrt{\frac{\det \mathcal{O}_{\mathrm{FV}}}{\det' \mathcal{O}}}$$

Pedagogical notes on functional determinants Dunne ’07

Functional determinant for radial operators

The bounce and the fluctuation operator is *O(4)* symmetric

$$\mathcal{O} = -\Box + V''(\overline{\varphi})$$

Just like hydrogen, we have the hyperspherical separation

Kleinert '04

$$\mathcal{O}_l \psi_l = -\ddot{\psi}_l - \frac{3}{\rho}\dot{\psi}_l + \frac{l(l+2)}{\rho^2}\psi_l + V''(\overline{\varphi})\,\psi_l \qquad \deg = (l+1)^2$$

Gel'fand Yaglom theorem

Gel'fand, Yaglom '59

Instead of multiplying the λ_n, one simply computes $\mathcal{O}_l \psi_l = 0$

$$\prod_n \lambda_n = \psi_l(\infty)$$

$$\ln\left(\frac{\det \mathcal{O}}{\det \mathcal{O}^{\mathrm{FV}}}\right) = \sum_{l=0}^{\infty} (l+1)^2 \ln \mathcal{R}_l(\infty)\,, \qquad \mathcal{R}_l \equiv \frac{\psi_l}{\psi_l^{\mathrm{FV}}}$$

$$\mathcal{O}_l\psi_l = -\ddot{\psi}_l - \frac{3}{\rho}\dot{\psi}_l + \frac{l(l+2)}{\rho^2}\psi_l + V''(\overline{\varphi})\,\psi_l$$

$\rho \to \infty$

$$\psi_l \to \psi_l^{\mathrm{FV}} \qquad \mathcal{R}_l \to 1 + \mathcal{O}\left(l^{-1}\right)$$

In *D>1* the functional determinant diverges quadratically, linearly and log

$$\Sigma = \ln\left(\frac{\det \mathcal{O}}{\det \mathcal{O}^{\mathrm{FV}}}\right) = \sum_{l=0}^{\infty}(l+1)^2 \ln \mathcal{R}_l(\infty) \to \sum_{l=0}^{\infty}(l+1)^2 \ln\left(1 + \frac{c}{l} + \ldots\right)$$

To regularize the sum, subtract the leading order divergencies…

$$\Sigma_f = \Sigma - \Sigma_a$$

…and then add them back with a renormalization procedure

Feynman diagrams = Effective action in Euclidean space-time

Baacke, Lavrelashvili ’04,
…
Andreassen, Frost, Schwartz ‘17

Zeta functions formalism

Minakshisundaram ’46, Hawking ’76, Elizalde ’94

Analytic results

Hard - even in the thin wall

Russians to the rescue…

Konoplich '87

> "...Even in this special case, we are not able to obtain a closed-form expression for A; we are stymied by an obdurate functional determinant."
> Callan, Coleman '77

The Standard model a ton of papers — Frampton '76 … '20

single unstable quartic, classically scale invariant — Isidori, Ridolfi, Strumia, '01

proper treatment of scale and gauge invariance — Andreassen, Frost, Schwartz '17

$$V(\varphi) = \frac{\lambda}{4}\phi^4$$

$$\overline{\varphi} = \sqrt{\frac{8}{\lambda}}\frac{R}{R^2-\rho^2} \qquad \mathcal{S}_0 = \frac{8\pi^2}{3\lambda} \qquad \mathcal{R}_l(\infty) = \frac{l(l-1)}{(l+2)(l+3)}$$

Quartic-quartic

add another segment and still come through analytically

MN, Guada '20

$$V = \frac{1}{4}\left(\lambda_2 v_2^4 - \lambda_1 v_1^4 + \lambda_1 \left(\varphi + v_1\right)^4\right) H(-\varphi) + \frac{\lambda_2}{4}\left(\varphi - v_2\right)^4 H(\varphi)$$

Two dimensionless parameters

$$x = \frac{v_1}{v_2}, \quad y = \frac{\lambda_1}{\lambda_2}$$

$$\overline{\varphi} = \sum_{s=1}^{2}\left((-1)^s v_s + \sqrt{\frac{8}{\lambda_s}}\frac{R_s}{R_s^2 - \rho^2}\right) H\left((-1)^s(\rho - R_T)\right)$$

dimension set by *v*

$$R_{1,2,T} = \frac{2}{v}\sqrt{\frac{2}{\lambda}}\,\frac{1+x}{x^4 y - 1}\left\{x^2\sqrt{y}, 1, \sqrt{\frac{x + x^4 y}{1+x}}\right\}$$

diverges in TW

$$\mathcal{S}_0 = \left(\frac{8\pi^2}{3\lambda}\right)\frac{1 + y + x^3 y\left(4 + xy\left(-3 + 6x^2 + (3+4x)x^4 y\right)\right)}{y(x^4 y - 1)^3}$$

the / zero disappears

The bounce is done, let's do the fluctuations

Quartic-quartic

MN, Guada '20

$$\mathcal{O}_l \psi_l = -\ddot{\psi}_l - \frac{3}{\rho}\dot{\psi}_l + \frac{l(l+2)}{\rho^2}\psi_l + V''\left(\overline{\varphi}\right)\psi_l = 0$$

False vacuum easy $\quad V''_{\rm FV} = 0, \quad \psi_l^{\rm FV} = \rho^l$

General solution

$$\psi_{ls} = A_{ls}\frac{\rho^l R_s^4}{(R_s^2-\rho^2)^2}\left(1 - 2\left(\frac{l-1}{l+2}\right)\frac{\rho^2}{R_s^2} + \frac{l(l-1)}{(l+2)(l+3)}\frac{\rho^4}{R_s^4}\right)$$
$$+ B_{ls}\frac{R_s^{l+4}}{(R_s^2-\rho^2)^2}\frac{R_s^{l+2}}{\rho^{l+2}}\left(1 - 2\left(\frac{l+3}{l}\right)\frac{\rho^2}{R_s^2} + \frac{(l+2)(l+3)}{l(l-1)}\frac{\rho^4}{R_s^4}\right)$$

δ

$$\psi_{l1} = \psi_{l2}\,, \qquad \dot{\psi}_{l1} = \dot{\psi}_{l2} + \mu_V\psi_{l1}\,, \qquad \mu_V = \frac{\lambda_1 v_1^3 + \lambda_2 v_2^3}{\dot{\overline{\varphi}}(R_T)}$$

$$\mathcal{R}_l(\infty) = \frac{(l-1)(l^3 + c_2 l^2 + c_1 l + c_0)}{(l+1)(l+2)^2(l+3)}$$

the $l=0$ zero disappears

$$c_0 = \frac{12(1+x)^2 x^4 y (1+x^3 y)^2}{(x^4 y - 1)^3}\,, \quad c_1 = \frac{2x\left(1+(1+2x)\,x^2 y\right)\left(2+3x+(3+4x)x^3 y\right)}{(x^4 y-1)^2}\,, \quad c_2 = \frac{1+4x+(4+7x)x}{x^4 y - 1}$$

Zero removal each symmetry gives a zero = Noether

MN, Guada '20

$$\mathcal{R}_l(\infty) = \frac{l(l-1)}{(l+2)(l+3)}$$

$$\mathcal{R}_l(\infty) = \frac{(l-1)(l^3 + c_2 l^2 + c_1 l + c_0)}{(l+1)(l+2)^2(l+3)}$$

$$\deg = (l+1)^2 \Rightarrow \begin{cases} 1\times \quad l=0 \\ 4\times \quad l=1 \end{cases}$$

classical scale invariance

translational invariance of $\overline{\varphi}(\rho)$ in *D=4*

need to remove the *l=1* zero eigenvalue, easy with explicit γ_n

$$\mathcal{O}_l \psi_l = \gamma_n \psi_l \implies \mathcal{R}_1'(\infty) = \frac{\prod_{n=2}^{\infty} \gamma_n}{\prod_{n=1}^{\infty} \gamma_n^{\mathrm{FV}}}\,.$$

more involved with Gel'fand-Yaglom, consider an off-set $\left(\mathcal{O}_1 + \mu_\varepsilon^2\right)\psi_1^\varepsilon = 0$

$$\mathcal{R}_1^\varepsilon(\infty) = \frac{\psi_1^\varepsilon(\infty)}{\psi_1^{\mathrm{FV}}(\infty)} \simeq \frac{\left(\mu_\varepsilon^2 + \gamma_1\right)\prod_{n=2}^{\infty}\gamma_n}{\prod_{n=1}^{\infty}\gamma_n^{\mathrm{FV}}} = \mu_\varepsilon^2 \mathcal{R}_1'(\infty)$$

$$\mathcal{R}_1'(\infty) = \lim_{\mu_\varepsilon^2 \to 0} \frac{1}{\mu_\varepsilon^2}\mathcal{R}_1^\varepsilon(\infty)$$

Zero removal

$$\mathcal{R}_1'(\infty) = \lim_{\mu_\varepsilon^2 \to 0} \frac{1}{\mu_\varepsilon^2} \mathcal{R}_1^\varepsilon(\infty)$$

perturbative expansion $\quad \psi_1^\varepsilon \simeq \psi_1 + \mu_\varepsilon^2\, \delta\psi_1$

$l = 1$

$$\left(\mathcal{O}_1 + \mu_\varepsilon^2\right)\left(\psi_1 + \mu_\varepsilon^2 \delta\psi_1\right) \simeq \mathcal{O}_1 \psi_1 + \mu_\varepsilon^2 \left(\psi_1 + \mathcal{O}_1 \delta\psi_1\right) = 0$$

$$\psi_{1s} = A_{1s} \frac{R_s^4 \rho}{\left(R_s^2 - \rho^2\right)^2}, \quad A_{11} = 1, \quad A_{12} = x^6 y^2$$

$$\delta\psi_{1s} = \frac{3R_s^6 \rho}{4\left(R_s^2 - \rho^2\right)^2} \left(\delta A_{1s} + \frac{\delta B_{1s}}{18} \left(\frac{\rho^4}{R_s^4} - 8\frac{\rho^2}{R_s^2} + 24 \log\rho + 8\frac{R_s^2}{\rho^2} - \frac{R_s^4}{\rho^4} \right) - \frac{A_{1s}}{18} \left(6\frac{\rho^2}{R_s^2} - 18 - 24\log\rho - \frac{R_s^2}{\rho^2} + \frac{R_s^4}{\rho^4} \right) \right)$$

Final product of *l=1* eigenvalues with zero removed

$$\mathcal{R}_1'(\infty) = \lim_{\mu_\varepsilon^2 \to 0} \frac{1}{\mu_\varepsilon^2} \frac{\psi_1 + \mu_\varepsilon^2 \delta\psi_1}{\psi_{\mathrm{FV}1}} \bigg|_{\rho=\infty} = \frac{\delta\psi_1}{\psi_{\mathrm{FV}1}} \bigg|_{\rho=\infty} = \frac{R_2^2}{24} \delta B_{12} = \frac{R_2^2}{24} \left(\frac{3\lambda}{8\pi^2} \right) \mathcal{S}_0 x^6 y^2$$

Finite sum

$$\mathcal{R}_l(\infty) = \frac{(l-1)(l^3 + c_2 l^2 + c_1 l + c_0)}{(l+1)(l+2)^2(l+3)}$$

MN, Guada '20

Four $l=1$ zeroes and negative $l=0$ $\quad \mathcal{R}_0(\infty) = -\frac{c_0}{12} < 0$

Expand up to ν^{-3}, $\nu = l + 1$ enough to remove all the infinities

now the sum is finite $\quad \Sigma_f = \sum_{\nu=1}^{\infty} \nu^2 \left(\ln \mathcal{R}_l(\infty) - \ln \mathcal{R}_l^a(\infty) \right)$

zeroes of the numerator

It can be computed in some generality, let $\quad \mathcal{R}_l(\infty) = \prod_{i=1}^{n} \frac{l+1-a_i}{l+1-b_i}$

$$\Sigma_f = \sum_{i=1}^{n} \left(\frac{a_i^3}{3}\gamma_E - \frac{a_i}{12}\left(1 + 3a_i - 6a_i^2\right) - \zeta_R'\left(-2, 3-a_i\right) - 2a_i\zeta_R'\left(-1, 3-a_i\right) \right.$$
$$\left. - a_i^2\zeta_R'\left(0, 3-a_i\right) - (a \to b) \right) + \ln \mathcal{R}_0(\infty) + 4\ln \mathcal{R}_1'(\infty)\,.$$

Renormalization via the zeta function

MN, Guada '20

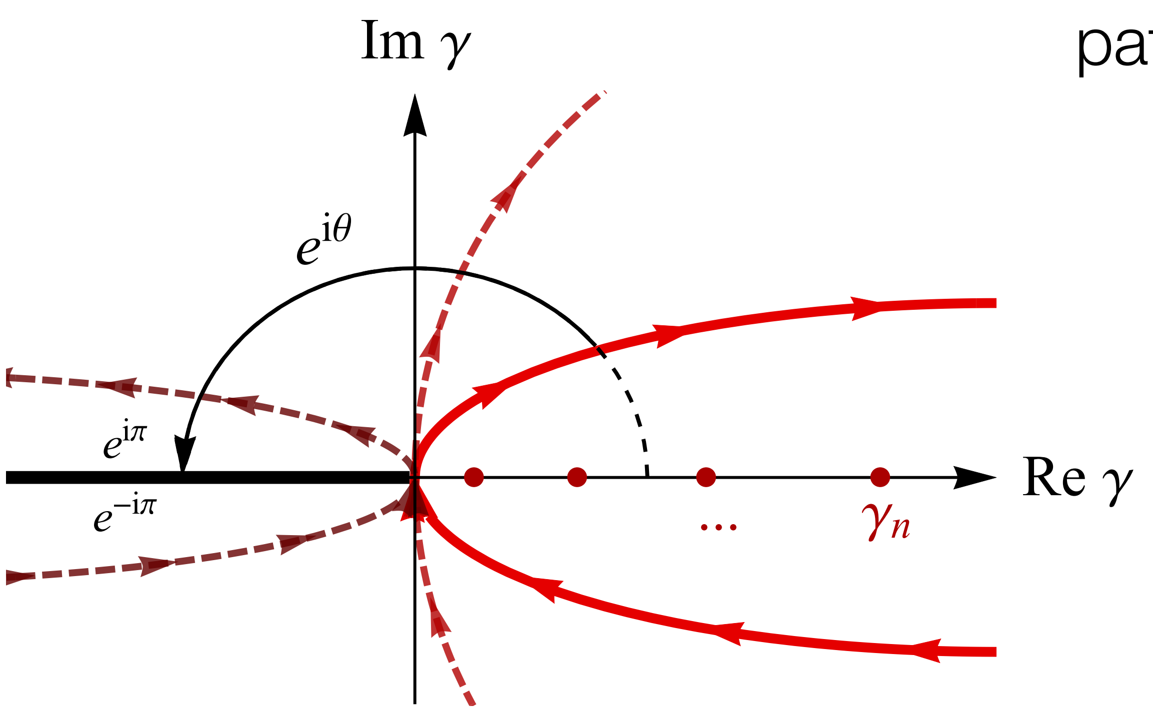
$$\ln\det\mathcal{O} = \sum_{\mathbf{n}} \ln\gamma_{\mathbf{n}} = -\frac{\mathrm{d}}{\mathrm{d}s}\sum_{\mathbf{n}}\left(\frac{\mu^2}{\gamma_{\mathbf{n}}}\right)^s\Bigg|_{s=0} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\mu^{2s}\zeta_{\mathcal{O}}(s)\right)\Big|_{s=0}$$

use the contour integral over a complex path to cover all the eigenvalues

$$\zeta_{\mathcal{O}} = \sum_{\mathbf{n}}\frac{1}{\gamma_{\mathbf{n}}^s} = \frac{1}{2\pi i}\oint\frac{\mathrm{d}\gamma}{\gamma^s}\frac{\mathrm{d}}{\mathrm{d}\gamma}\ln\psi(\infty,\gamma)$$

Cauchy

The sum above converges for *s>D/2*, however to get to *s=0* we split the path and deform upper and lower branch



$$\zeta_{\mathcal{O}} = \frac{\sin\pi s}{\pi}\int_0^\infty\frac{\mathrm{d}\gamma}{\gamma^s}\frac{\mathrm{d}}{\mathrm{d}\gamma}\ln\psi(\infty,-\gamma)$$

False vacuum expansion

MN, Guada '20

$$\zeta = \frac{\sin \pi s}{\pi} \mu^{2s} \sum_\nu \nu^2 \int_0^\infty \frac{d\gamma}{\gamma^s} \frac{d}{d\gamma} \ln \left(\frac{\psi_l(\infty, -\gamma)}{\psi_l^{FV}(\infty, -\gamma)} \right)$$

comes from

$$\mathcal{O}_l \psi_l(\rho, \gamma) = \gamma\, \psi_l(\rho, \gamma)$$

easy for FV $\quad \psi_l^{FV}(\rho, -\gamma) = I_\nu \left(\sqrt{\gamma}\rho\right) / \rho$

perturbative expansion around FV $\quad \psi_l(\rho, -\gamma) \simeq f_l\left(\gamma\right) \psi_l^{FV}(\rho, -\gamma)$

$$\psi_l(\rho, -\gamma) = \psi_l^{FV}(\rho, -\gamma) + \int_0^\rho d\rho_1 G(\rho, \rho_1) V''(\rho_1) \psi_l(\rho_1, -\gamma)$$

$$G(\rho, \rho_1) = \frac{\rho_1^2}{\rho} \left(I_\nu(\sqrt{\gamma}\rho) K_\nu(\sqrt{\gamma}\rho_1) - I_\nu(\sqrt{\gamma}\rho_1) K_\nu(\sqrt{\gamma}\rho) \right)$$

Asymptotic behaviour

MN, Guada '20

we're looking for the high-l expansion around the FV $\rho \sim \infty$

$$V''(\rho) = \sum_s V_s''(\rho) H\left((-1)^s(\rho - R_T)\right) - \mu_V \delta(\rho - R_T)$$

evaluating the FV expansion up to $\mathcal{O}\left(V''^3\right)$, we have

known Dunne, Min '05

$$\ln f_l^a = \sum_s \int_0^\infty \mathrm{d}\rho\rho V_s'' \left(\frac{t}{2\nu} + \frac{t^3}{16\nu^3}\left(1 - 6t^2 + 5t^4 - 2\rho^2 V_s''\right)\right) H\left((-1)^s\left(\rho - R_T\right)\right)$$
$$- \mu_V R_T \left(\frac{t}{2\nu} + \frac{t^3}{16\nu^3}\left(1 - 6t^2 + 5t^4\right) + \mu_V R_T \frac{t^2}{8\nu^2} + \left(\mu_V R_T\right)^2 \frac{t^3}{24\nu^3}\left(1 - \frac{3}{\mu_V^2}\left(V_1'' + V_2''\right)\right)\right)\Bigg|_{\rho=R_T}$$

New! Includes the delta function

very non-trivial cross-check $f_l^a(0) = \mathcal{R}_l^a(\infty)$

and by definition above $f_l(0) = \mathcal{R}_l(\infty)$

$$-\zeta_f'(0) = \sum_\nu \nu^2 \ln \frac{\mathcal{R}_l(\infty)}{f_l^a(0)} = \Sigma_f$$

Renormalization

MN, Guada '20

the integrals in $\ln f_l^a$ can now be done with

Backup slides on Gamma regulation

$$\frac{\sin \pi s}{\pi} \mu^{2s} \int_0^\infty \frac{\mathrm{d}\gamma}{\gamma^s} \frac{\mathrm{d}}{\mathrm{d}\gamma} t^n = -\frac{\Gamma\left(s + \frac{n}{2}\right) (\mu\rho)^{2s}}{\Gamma(s)\,\Gamma\left(\frac{n}{2}\right)} \nu^{-2s}$$

known Dunne, Min '05

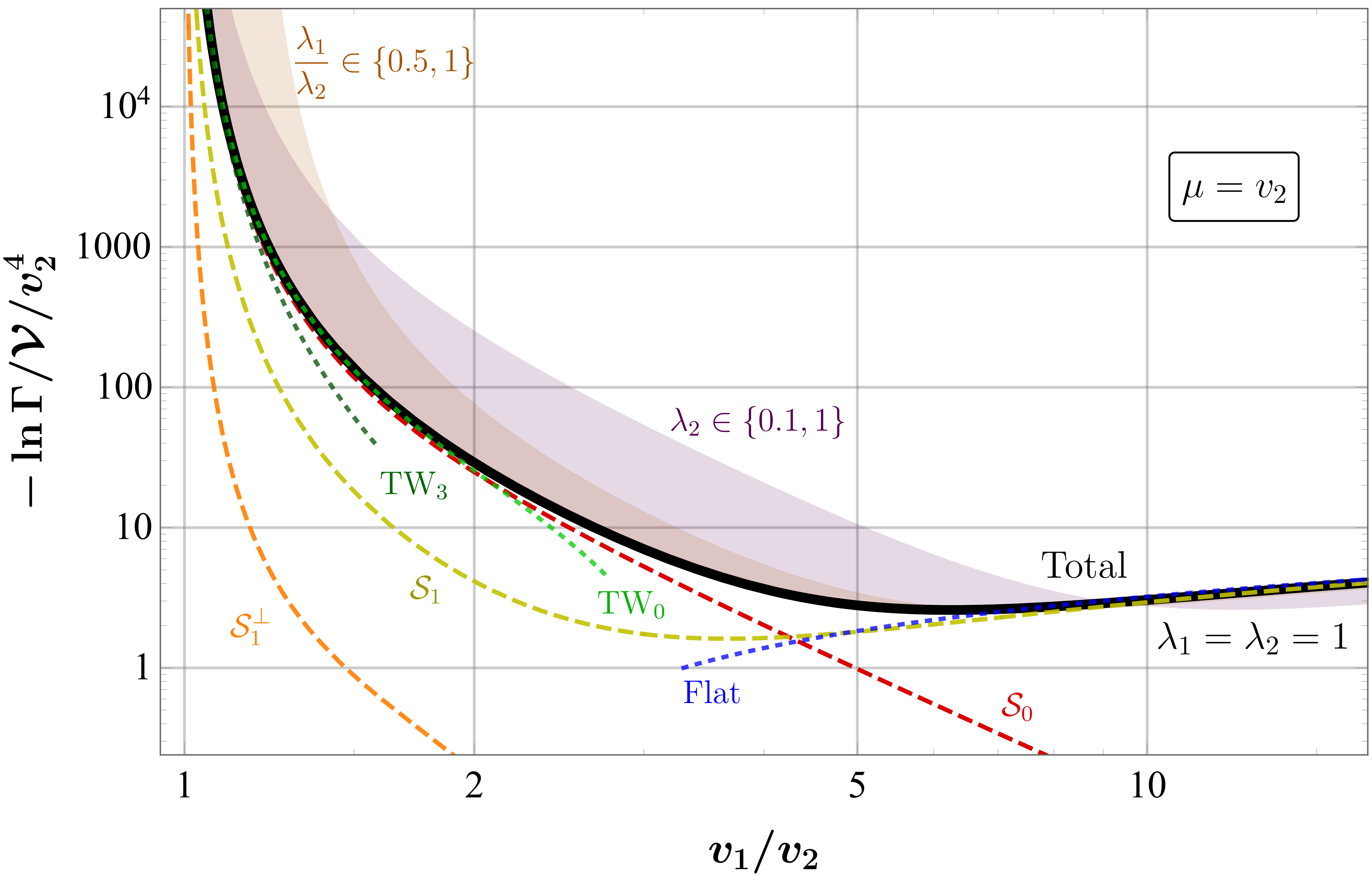
and finally, we get the log and R_T pieces

$$\zeta_a'(0) = \sum_s \frac{1}{8} \int_0^\infty \mathrm{d}\rho\, \rho^3 V_s''^2 \left(\ln\left(\frac{\mu\rho}{2}\right) + \gamma_E + 1\right) H\left((-1)^s (\rho - R_T)\right)$$
$$- \left[\frac{(\mu_V R_T)^2}{16} + \frac{(\mu_V R_T)^3}{24}\left(1 - \frac{3}{\mu_V^2}\left(V_1'' + V_2''\right)\Big|_{R_T}\right)\right]\left(\ln\left(\frac{\mu R_T}{2}\right) + \gamma_E + 1\right)$$

New! Includes the delta function

all the integrals analytic, `Mathematica` notebook in supplementary

MN, Guada '20



$$-\ln\frac{\Gamma}{\mathcal{V}}\frac{1}{v^4} \simeq \begin{cases} \frac{1}{\varepsilon^3}\left(\frac{2\pi^2}{3\lambda}+\frac{2}{9}+\frac{\pi}{2\sqrt{3}}-\frac{1}{12}\ln\frac{2\lambda v^2}{\mu^2}\right), & \mathrm{TW} \\ \frac{7}{12}-2\zeta_R'(-1)+\frac{1}{3}\ln\frac{y^2\lambda^2 v^4 x^6}{32\pi^3\mu^4}, & \mathrm{Flat} \end{cases}$$

Outlook

False vacuum decay

include curvature: polygonal vs. Coleman - de Luccia

universal prefactor from extended polygonal - wip

multifield prefactor

Phenomenological applications

gravitational waves in physical settings

wall velocity @ NLO

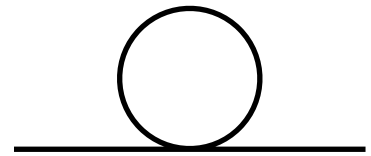
baryogenesis - transport equations

Extras

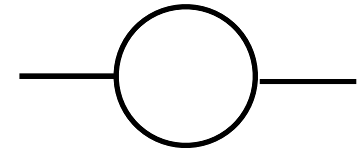
Renormalization via zeta assigning infinities in QFT

Feynman diagrams

$n = 1$



$n = 2$



any QFT book

$$\int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma\left(n - \frac{d}{2}\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2}} \to \frac{i}{(4\pi)^2} \begin{cases} -\Delta\left(\frac{2}{\varepsilon} - \log\Delta - \gamma_E + \log 4\pi\right), & n = 1 \\ \frac{2}{\varepsilon} - \log\Delta - \gamma_E + \log 4\pi, & n = 2 \end{cases}$$

Analytical continuation of zeta

Dunne '09

$$\zeta_{\mathrm{R}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \Gamma(s) := \int_0^\infty \mathrm{d}t\, t^{s-1} e^{-t}$$

$\Re(s) > 1$

$$\zeta_{\mathrm{R}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} \sum_{n=1}^{\infty} e^{-nt} = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} \frac{1}{e^t - 1}$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} \frac{e^{-t/2}}{2\sinh(t/2)}$$

Analytical continuation of zeta

Dunne '09

$\Re(s) > 1$

$$\zeta_{\rm R}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} \sum_{n=1}^{\infty} e^{-nt} = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} \frac{1}{e^t - 1}$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} \frac{e^{-t/2}}{2\sinh(t/2)}$$

Remove the divergent piece(s)

$s \to 0$

$$\zeta_{\rm R}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} \left[\frac{e^{-t/2}}{2\sinh(t/2)} - \frac{1}{t} \right] + \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-2} e^{-t/2}$$

$$= \underbrace{\frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} \left[\frac{e^{-t/2}}{2\sinh(t/2)} - \frac{1}{t} \right]}_{\to 0} + \frac{2^{s-1}}{s-1}$$

Mathematica

$$\zeta_{\rm R}(0) = -\frac{1}{2} \qquad \zeta_{\rm R}'(0) = -\frac{1}{2}\ln(2\pi)$$

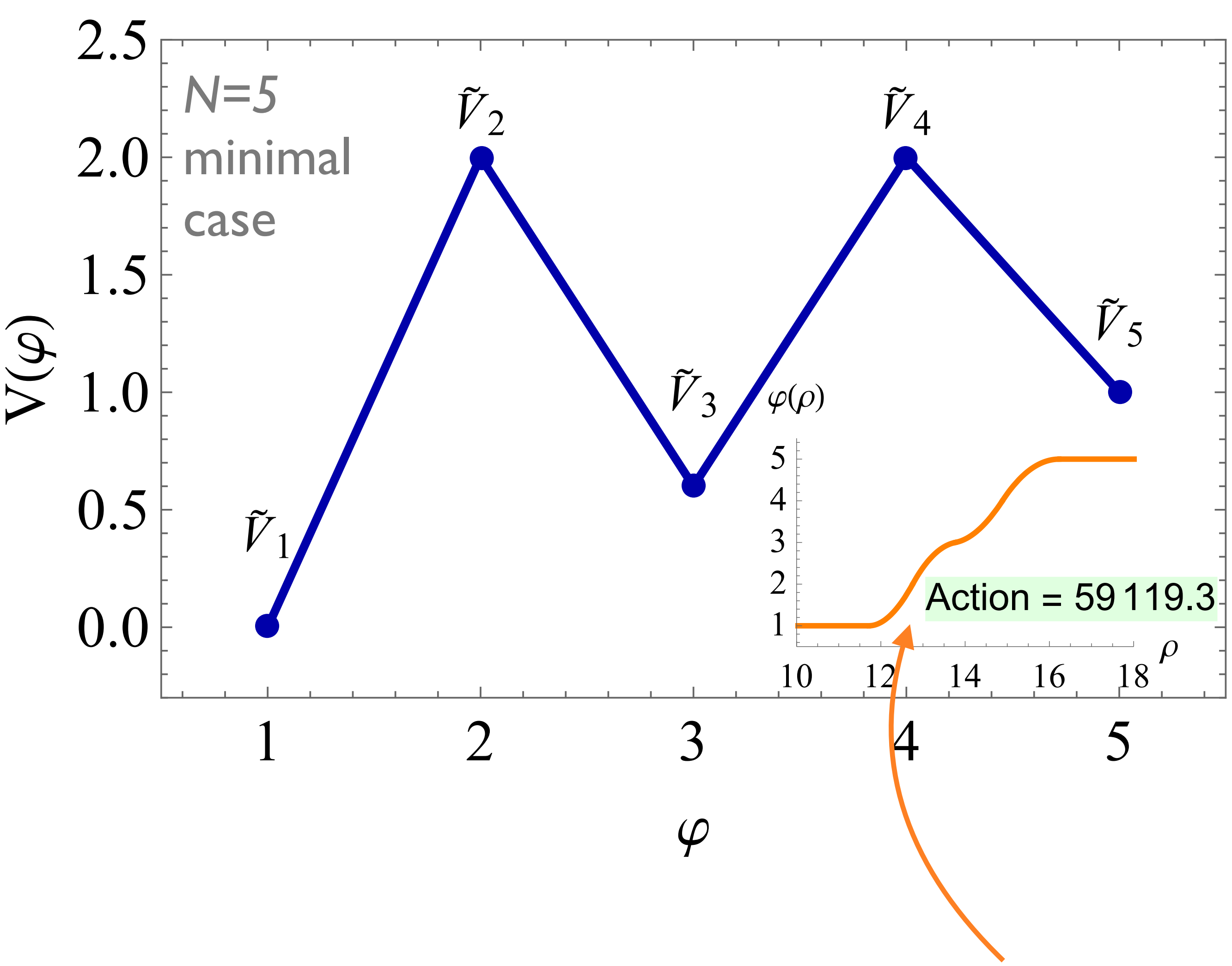
Disappearing instanton

Intermediate minima, multi-step transitions

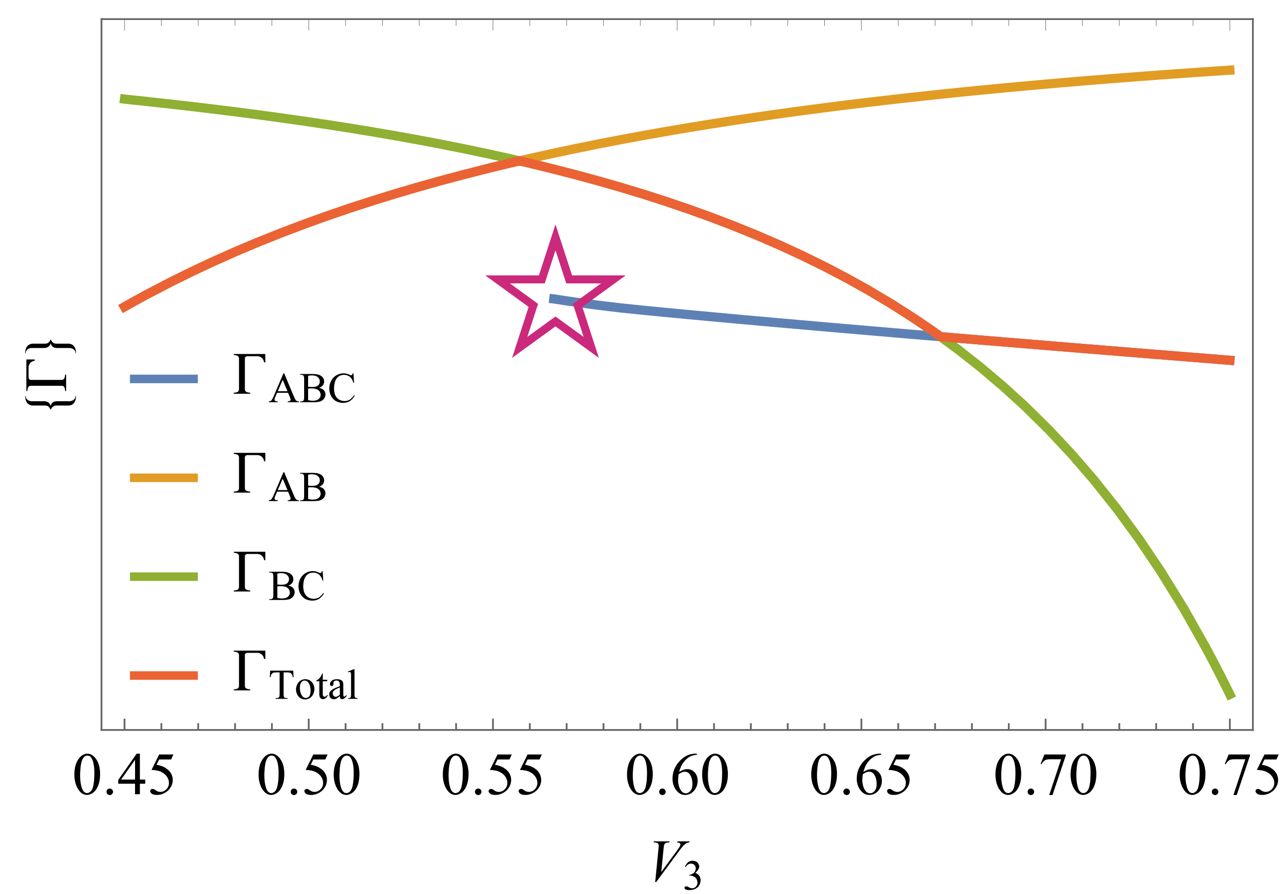
Dahlen, Brown '11

appear in theories with many fields, relaxion-type potentials

Q: which transition wins, direct tunneling or two subsequent transitions?

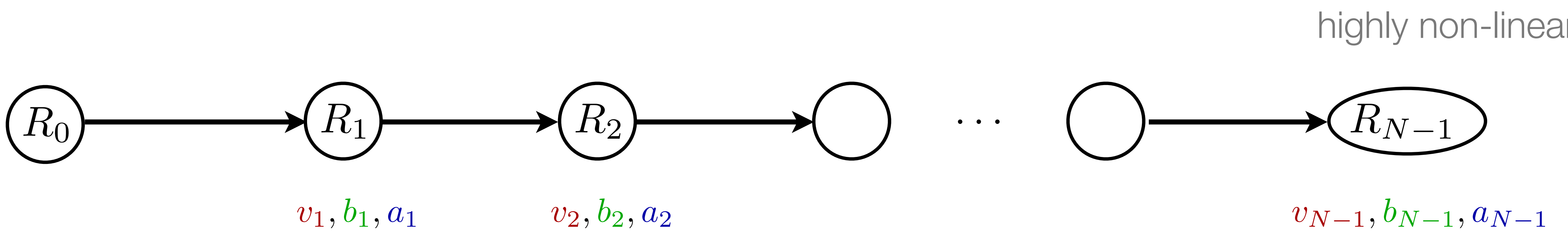


bubble with two walls



Direct tunneling impossible when intermediate minimum too low

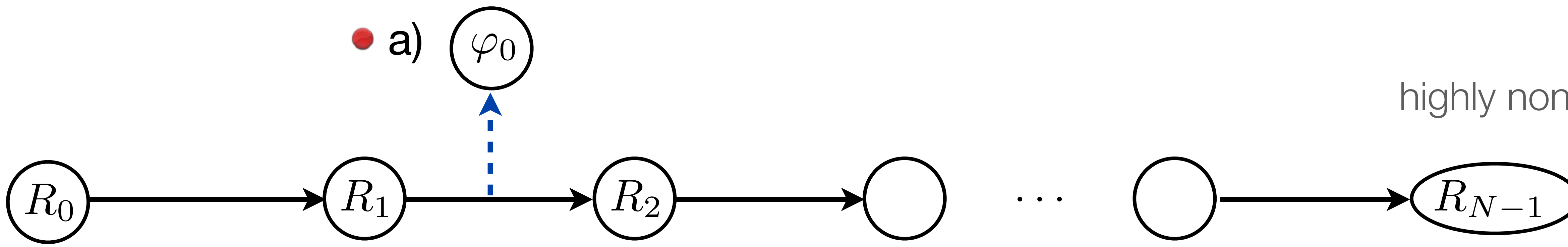
- b) $\varphi_0 = \tilde{\varphi}_1$



Matching

$$a_1 R_0^D + \sum_{i=1}^{N-2} \left(a_{i+1} - a_i\right) R_i^D - a_{N-1} R_{N-1}^D = 0$$

- a)
- b)

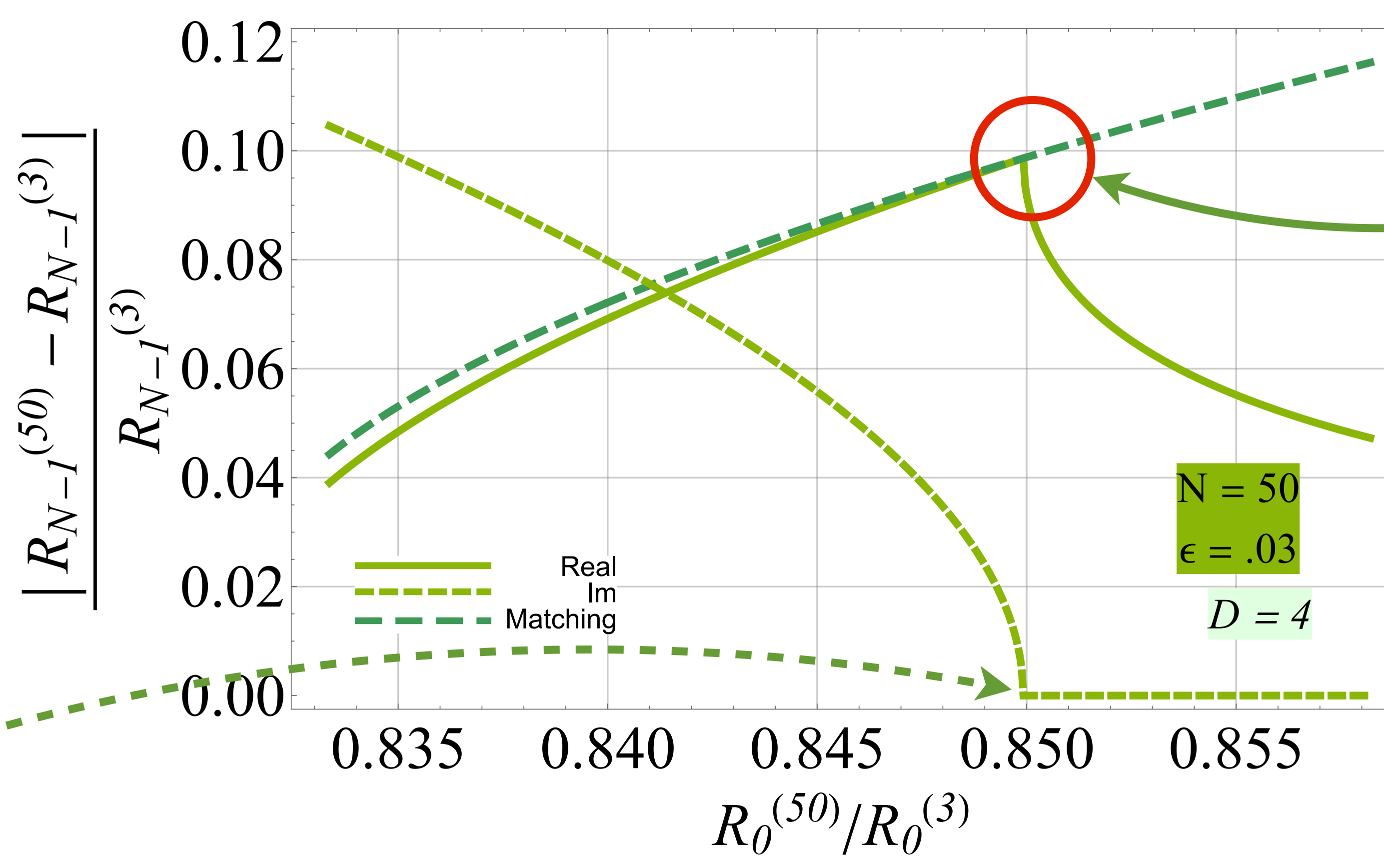


highly non-linear

Matching

$$a_1 R_0^D + \sum_{i=1}^{N-2} \left(a_{i+1} - a_i\right) R_i^D - a_{N-1} R_{N-1}^D = 0$$

case b)



bounce solution

stiff behaviour

becomes imaginary, coincides with the solution

Rescaling

$R_0 = 0$ $\quad$ $R_1 = 0$

Use Derrick's theorem to find the solution

estimate the initial value of R_i

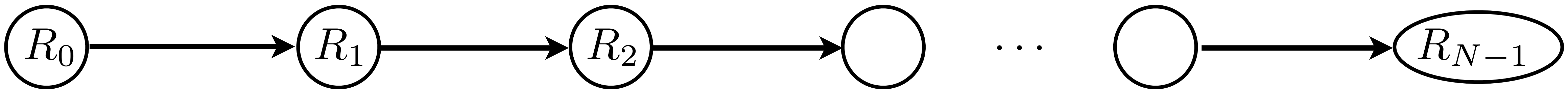
compute the kinetic $\mathcal{T}(R_i)$ and potential pieces $\mathcal{V}(R_i)$

rescale the radius R_i by $\lambda = \sqrt{\frac{(2-D)\mathcal{V}(R_i)}{D\mathcal{V}(R_i)}}$

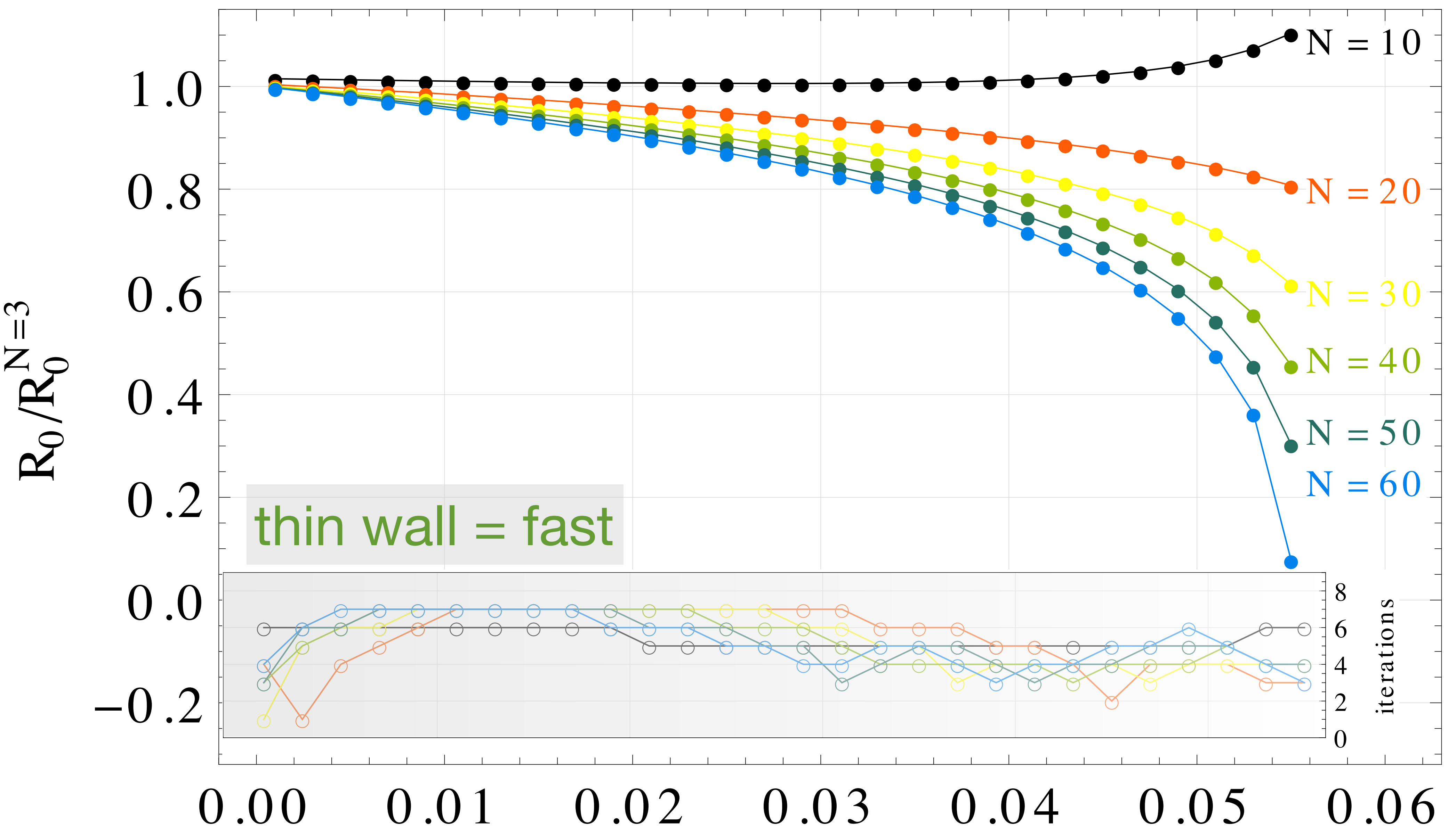
iterate until $|\lambda - 1| \lesssim \frac{1}{N}$

retrieve φ_0 from $R_i(\varphi_0) = \sqrt{\frac{\tilde{\varphi}_{i+1} - \varphi_0}{a_i}}$

b)



Rescaling converged to permille level



~200 bounces in 2 secs

failing for small radii, expansion