HYDRODYNAMIC DIFFUSION AND ITS BREAKDOWN NEAR AdS_2 FIXED POINTS

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based on 2011.12301 (with Daniel Arean, Blaise Gouteraux, Kenta Suzuki)

INTRODUCTION

- QFT at non-zero temperature is important, but can be very complicated.

  Especially if there is no quasiparticle or other perturbative description.

- However, some aspects of the dynamics are relatively simple and general.

  Specifically, dynamics of conserved charge density operators over very long time and distance scales.

- These dynamics are governed by simple effective theories: **hydrodynamics**.

  When are these effective theories valid?

HYDRODYNAMICS

- Assumption: local thermal equilibrium is achieved over large enough scales.

$$t \gg \tau_{eq} \qquad\qquad x \gg l_{eq}$$

The system can be described by its conserved charge densities.

e.g. energy density $\varepsilon(t, x)$, momentum density $\overrightarrow{\Pi}(t, x)$, U(1) density $\rho(t, x)$, etc

- These each obey a local conservation equation: $\partial_t \varepsilon + \nabla \cdot j_\varepsilon = 0$ etc.

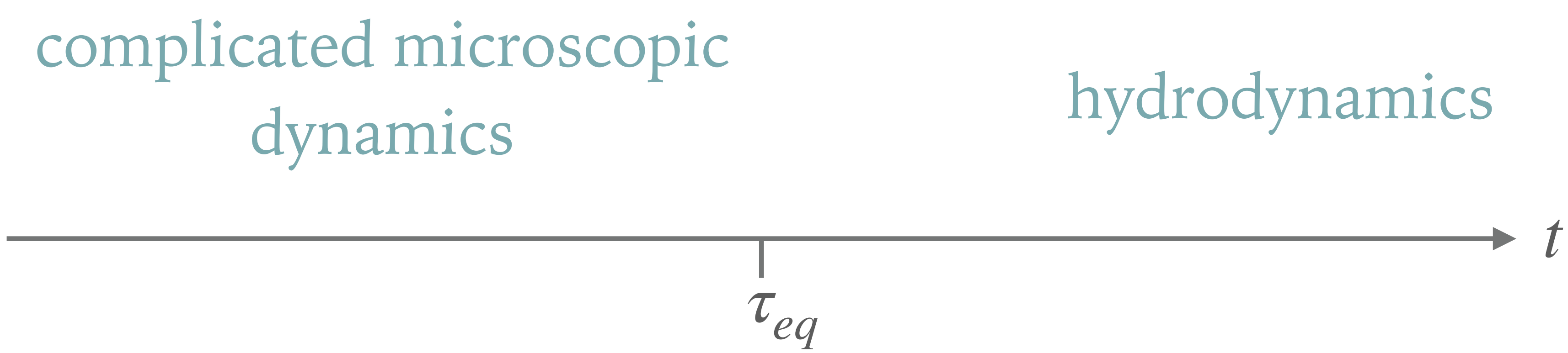
Currents are expressed as a gradient expansion of the densities

e.g. $$j_\varepsilon = -D\nabla\varepsilon - D_2\nabla^3\varepsilon - \ldots$$

$\longrightarrow$ Hydrodynamic equations e.g. $$\partial_t \varepsilon = D\nabla^2\varepsilon + D_2\nabla^4\varepsilon + \ldots$$

BREAKDOWN OF HYDRODYNAMICS

- Expect this description to break down at short scales



e.g. in a Fermi liquid $\tau_{qp} \sim 1/T^2$ and so expect $\tau_{eq} \sim 1/T^2$

- If there are no quasiparticles, we would perhaps expect $\tau_{eq} \sim 1/T$. Can we say anything more than this?

- This talk: In some cases, local equilibration scales τ_{eq} and l_{eq} are governed by basic low energy properties of the state.

HYDRODYNAMIC MODES

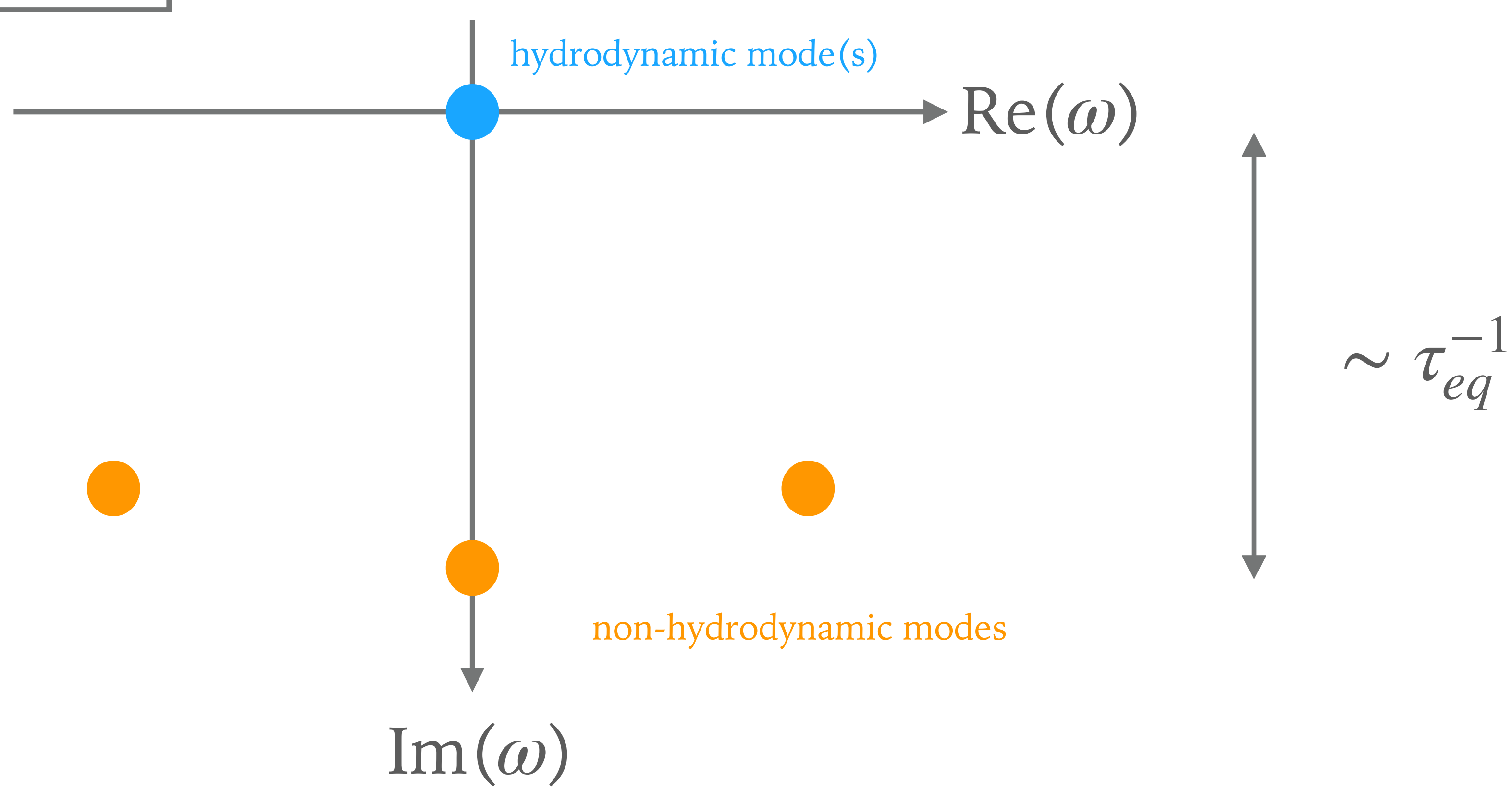
- One way to define τ_{eq} and l_{eq} is via properties of the hydrodynamic modes.

e.g. $\partial_t \varepsilon = D\nabla^2 \varepsilon + D_2 \nabla^4 \varepsilon + \ldots \longrightarrow \qquad \omega_{hydro}(k) = -iDk^2 - iD_2 k^4 + \ldots$

- Appear as poles of retarded Green's functions of conserved charge densities

$\boxed{k = 0}$

$$G^{-1}_{\varepsilon\varepsilon}(\omega, k) = 0$$

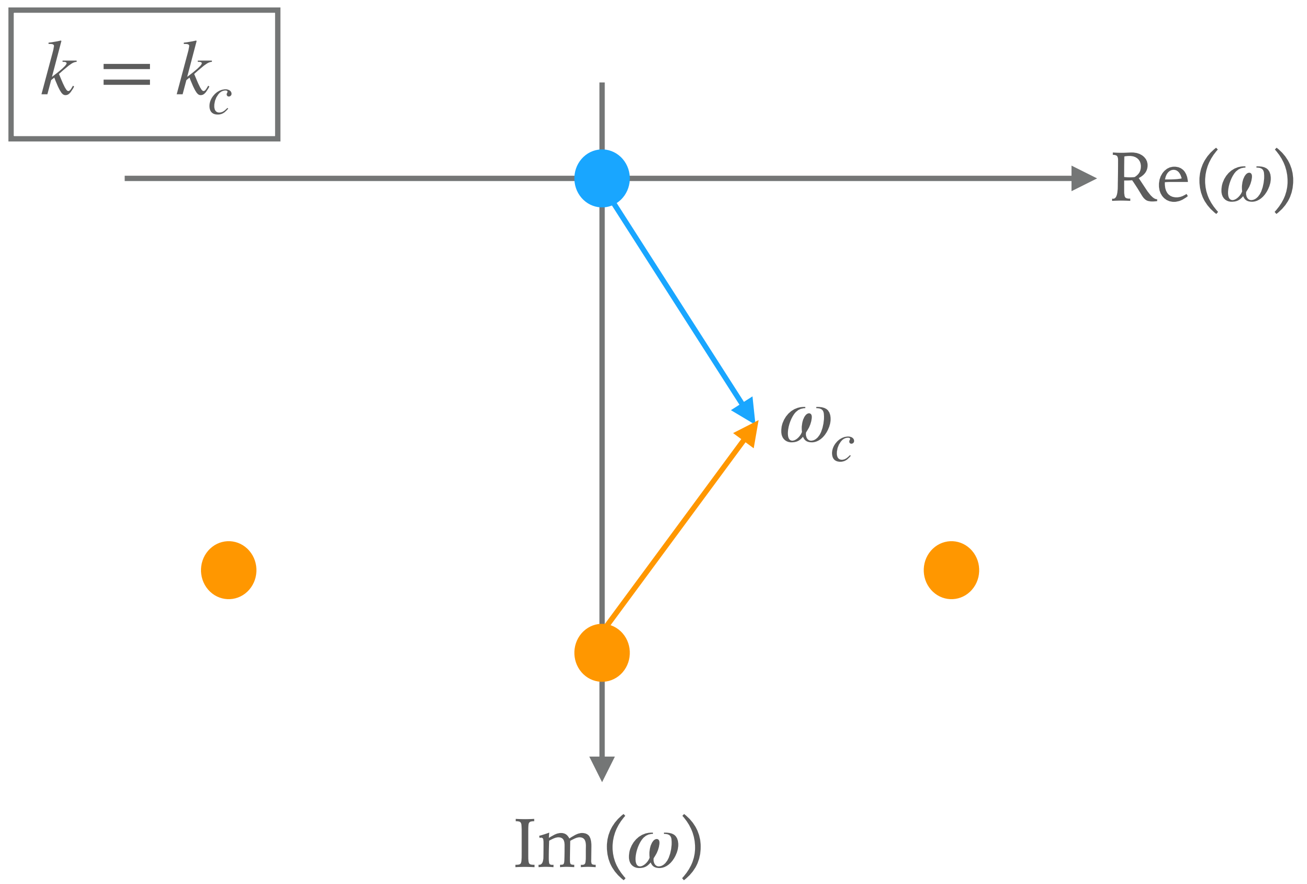


CONVERGENCE OF DISPERSION RELATIONS

- Another sensible definition: k_{eq} is radius of convergence of the series

$$\omega_{hydro}(k) = -i \sum_{i=1}^{\infty} D_n k^{2n}$$

- For complex k, the dispersion relations have branch points at $k = k_c$.
  Radius of convergence is set by closest branch point to the origin: $k_{eq} = |k_c|$.



The branch point appears as a collision of poles at (k_c, ω_c)

Grozdanov et al (1904.01018),
Grozdanov et al (1904.12862),
...

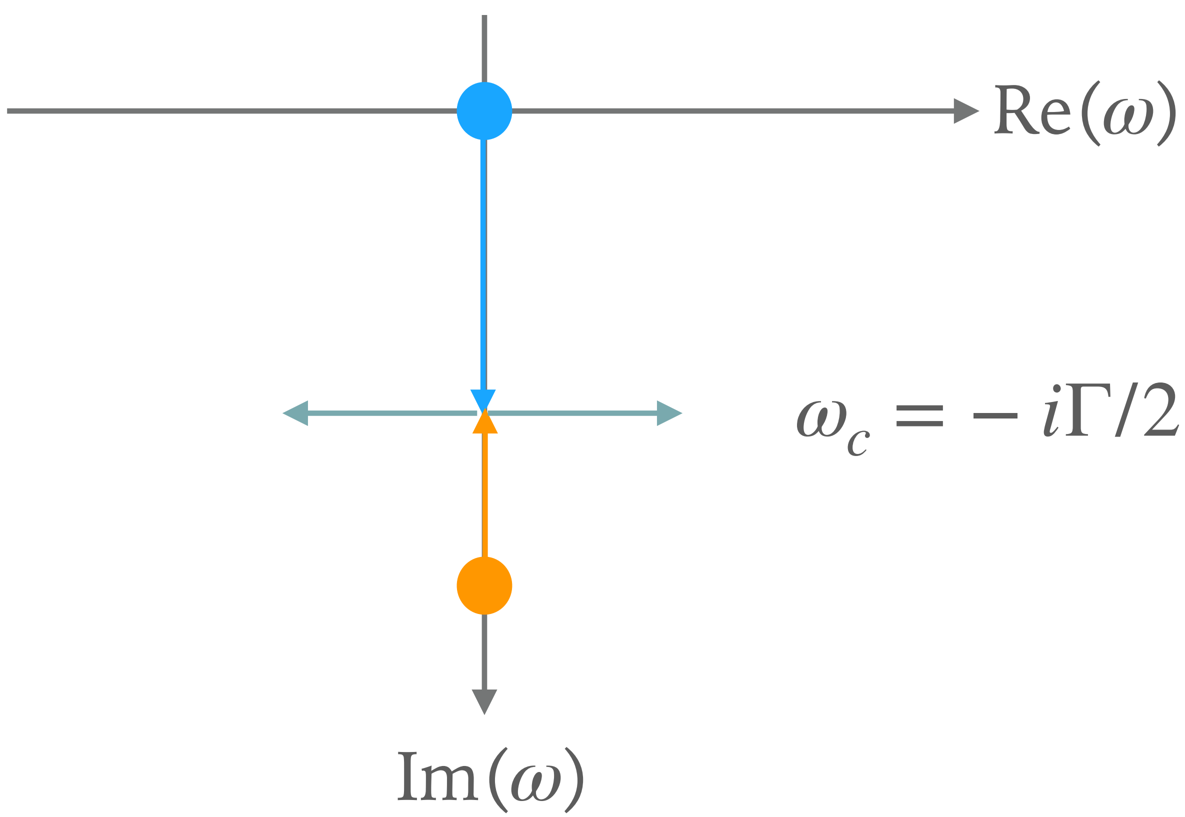
EXAMPLE

- As a simple example, consider $G^{-1}(\omega, k) = \omega^2 + i\omega\Gamma - v^2k^2$

  There are two poles with dispersion relations $\omega_{\pm}(k) = -\dfrac{i\Gamma}{2} \pm \sqrt{k^2v^2 - \dfrac{\Gamma^2}{4}}$

- One pole is hydrodynamic: $\omega_{hydro}(k) = -i\dfrac{v^2}{\Gamma}k^2 - i\dfrac{v^4}{\Gamma^3}k^4 - 2i\dfrac{v^6}{\Gamma^5}k^6 + \ldots$

  The branch point at $k^2 = k_c^2 = \Gamma^2/(4v^2)$ sets its radius of convergence.



SUMMARY

- I will use the location of pole collisions (k_c, ω_c) to define equilibration scales

$$l_{eq} = \frac{1}{k_{eq}} = \frac{1}{|k_c|} \qquad \text{and} \qquad \tau_{eq} = \frac{1}{\omega_{eq}} = \frac{1}{|\omega_c|}$$

- We examined states governed by an AdS_2 fixed point in the IR.
  In the low temperature limit

$$\omega_{eq} \to 2\pi T\Delta \qquad\qquad k_{eq}^2 \to \frac{\omega_{eq}}{D} = \frac{2\pi T\Delta}{D}$$

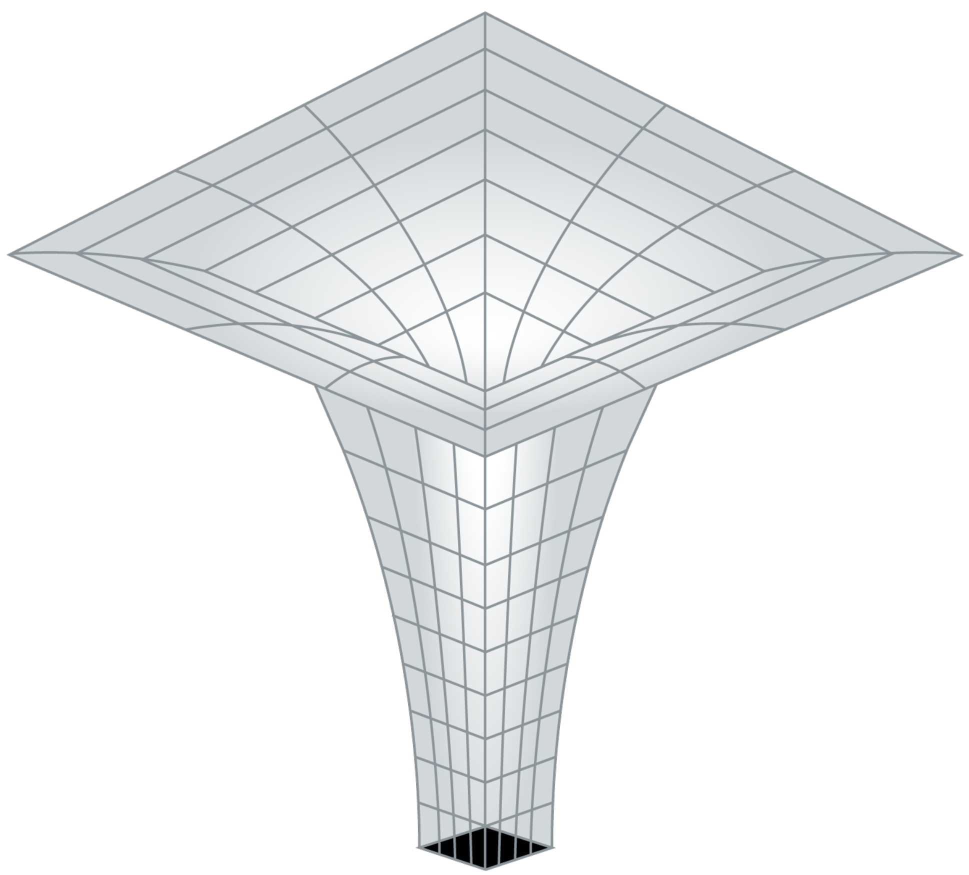
Δ : IR scaling dimension

D : hydrodynamic diffusivity

- An alternative way of looking at it: $D = \omega_{eq} k_{eq}^{-2}$.

AdS$_2$ FIXED POINTS

- Low temperature black brane solutions of $S = \int d^4x\sqrt{-g}\,(R - 2\Lambda + \mathscr{L}_{matter})$



UV: asymptotically AdS$_4$

RG flow depends on details

IR: AdS$_2$ x R^2

$$ds^2 = -\frac{r^2}{L^2}dt^2 + \frac{L^2dr^2}{r^2} + L_x^2(dx_1^2 + dx_2^2)$$

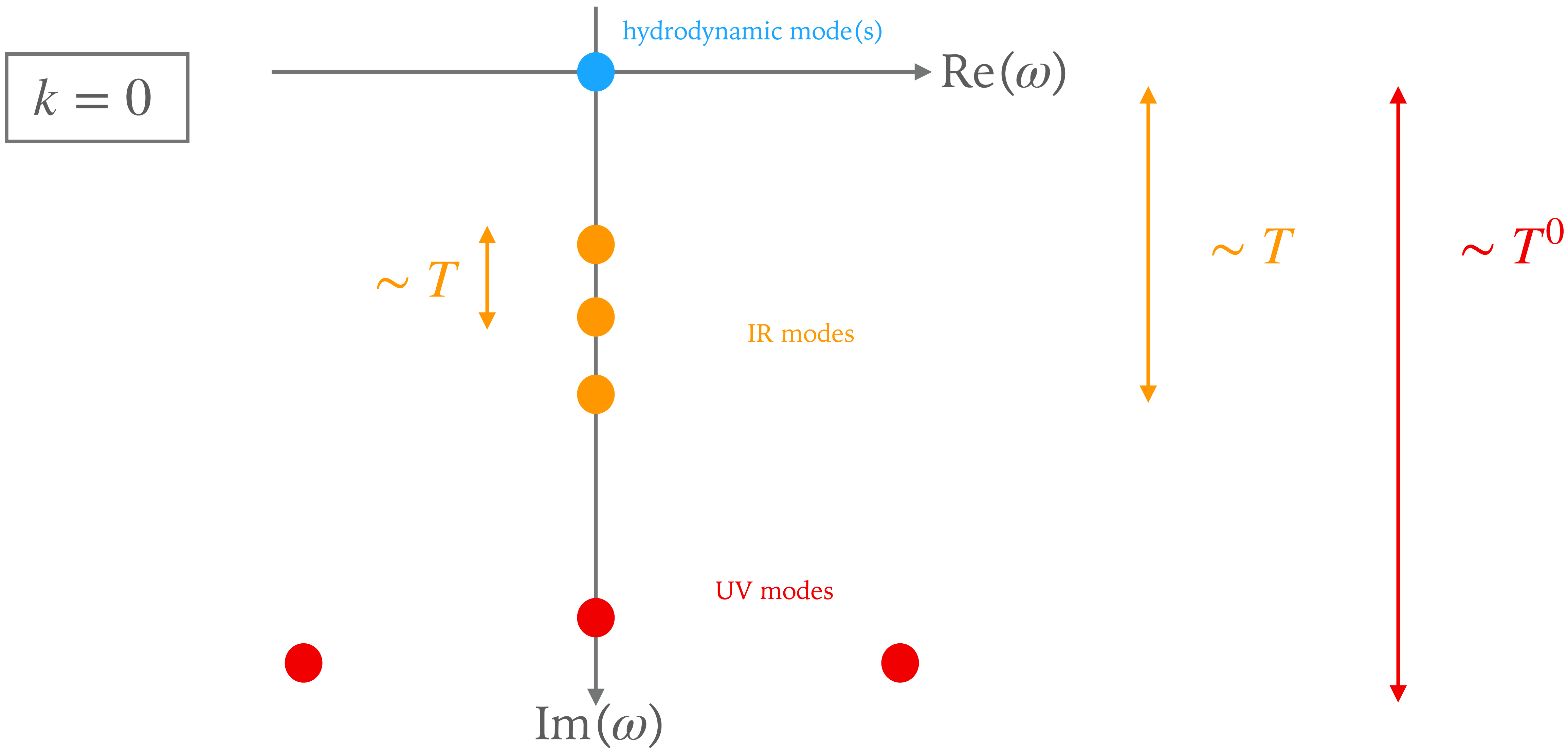
- Scaling symmetry in time: $t \to \lambda t$ and $r \to \lambda^{-1} r$.
- Similar IR fixed points arise in SYK models of strongly interacting fermions.

Maldacena, Stanford, Yang (1606.01847), ...

QUASINORMAL MODES

- Green's function poles correspond to quasinormal modes of the spacetime.
- There are three types:

Son, Starinets (hep-th/0205051)



- I will ignore the UV modes from now on.

THE HYDRODYNAMIC MODES

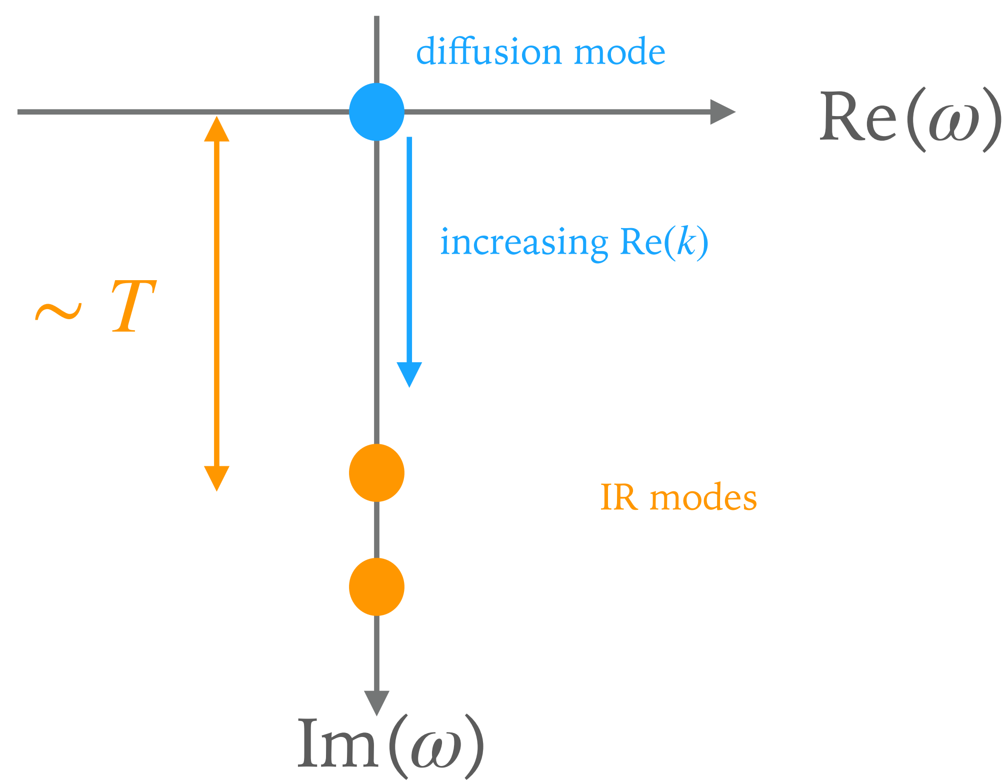
- The hydrodynamic mode(s) satisfy $\omega_{hydro}(k=0)=0$.

  Even with restrictions made so far, different states will have different

  * Number of hydrodynamic modes
  * Structure of dispersion relations: $\omega(k) = vk + \ldots$ or $\omega(k) = -iDk^2 + \ldots$
  * Hydrodynamic coefficients (e.g. speed v, diffusivity D etc.)

- I am only going to talk about hydrodynamic diffusion modes i.e.

$$\omega_{hydro}(k) = -iDk^2 + \ldots$$



EXAMPLES

- Simplest example: axion theory $\mathscr{L}_{matter} = -\frac{1}{2}\sum_{i=1}^{2}\partial_\mu\varphi_i\partial^\mu\varphi_i \qquad \varphi_i = k_L x^i$

  Andrade, Withers (1311.5157)

$$ds^2 = r^2\left(-f(r)dt^2 + dx_1^2 + dx_2^2\right) + \frac{dr^2}{r^2 f(r)} \qquad f(r) = 1 - \frac{k_L^2}{2r^2} - \left(1 - \frac{k_L^2}{2r_0^2}\right)\frac{r_0^3}{r^3}$$

  One hydrodynamic mode: diffusion of energy $\omega(k) = -iD_\varepsilon k^2 + \ldots$

- Next simplest: RN-AdS4 $\mathscr{L}_{matter} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad A_t = \mu\left(1 - \frac{r_0}{r}\right)$

$$f(r) = 1 - \left(1 + \frac{\mu^2}{4r_0^2}\right)\frac{r_0^3}{r^3} + \frac{\mu^2 r_0^2}{4r^4}$$

  Four hydrodynamic modes: two pressure waves $\omega(k) = \pm vk + \ldots$

  diffusion of temperature $\omega(k) = -iD_T k^2 + \ldots$

  diffusion of momentum $\omega(k) = -iD_\Pi k^2 + \ldots$

THE INFRA-RED MODES

- The IR modes are associated to the AdS_2 region of the spacetime.
- Cut off the rest of the spacetime and do holography in AdS_2.
  - ✳ Each IR operator has a dimension $\Delta(k)$ that governs its Green's function

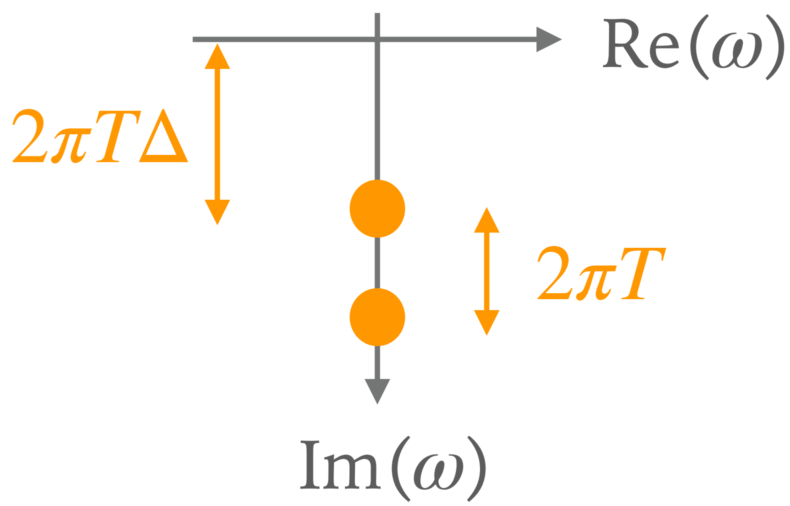
$$\mathscr{G}_{IR} \propto T^{2\Delta(k)-1} \frac{\Gamma\left(\frac{1}{2} - \Delta(k)\right)\Gamma\left(\Delta(k) - \frac{i\omega}{2\pi T}\right)}{\Gamma\left(\frac{1}{2} + \Delta(k)\right)\Gamma\left(1 - \Delta(k) - \frac{i\omega}{2\pi T}\right)}$$

Faulkner et al (0907.2694)

  - ✳ The AdS2 spacetime has quasinormal modes at

$$\omega_n(k) = -i2\pi T(n + \Delta(k))$$

$$n = 0,1,2,\ldots$$



- The full spacetime inherits these modes in the limit of small T and k.

EXAMPLES

- For the axion theory, the IR operator that couples to energy density ε has

$$\Delta(k) = \frac{1}{2}\left(1 + \sqrt{9 + 8k^2/k_L^2}\right) \longrightarrow \omega_n(0) = -\,i2\pi T(n+2)$$

- For RN-AdS4, the IR operators that couple to temperature T and momentum Π perturbations have

$$\Delta_T(k) = \frac{1}{2}\left(1 + \sqrt{5 + 8k^2/\mu^2 + 4\sqrt{1 + 4k^2/\mu^2}}\right) \longrightarrow \omega_n(0) = -\,i2\pi T(n+2)$$

$$\Delta_\Pi(k) = \frac{1}{2}\left(1 + \sqrt{5 + 8k^2/\mu^2 - 4\sqrt{1 + 4k^2/\mu^2}}\right) \longrightarrow \omega_n(0) = -\,i2\pi T(n+1)$$

Edalati et al (1005.4075)

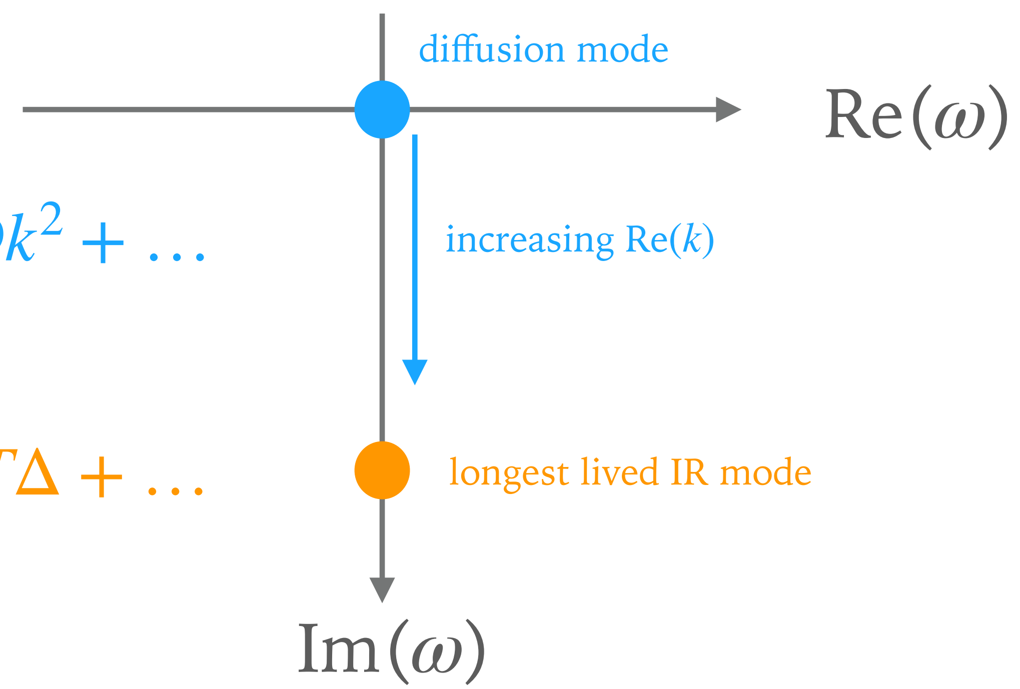
- More generally, expect a perturbation to couple to multiple IR operators.

CARTOON OF POLE COLLISION

- In all three cases, the two longest-lived modes are

$$\omega_{hydro}(k) = -iDk^2 + \ldots$$

$$\omega_0(k) = -i2\pi T\Delta + \ldots$$



- Be very naive: the modes will collide at

$$\omega_c = -i2\pi T\Delta \qquad \text{and} \qquad k_c^2 = \frac{2\pi T\Delta}{D}$$

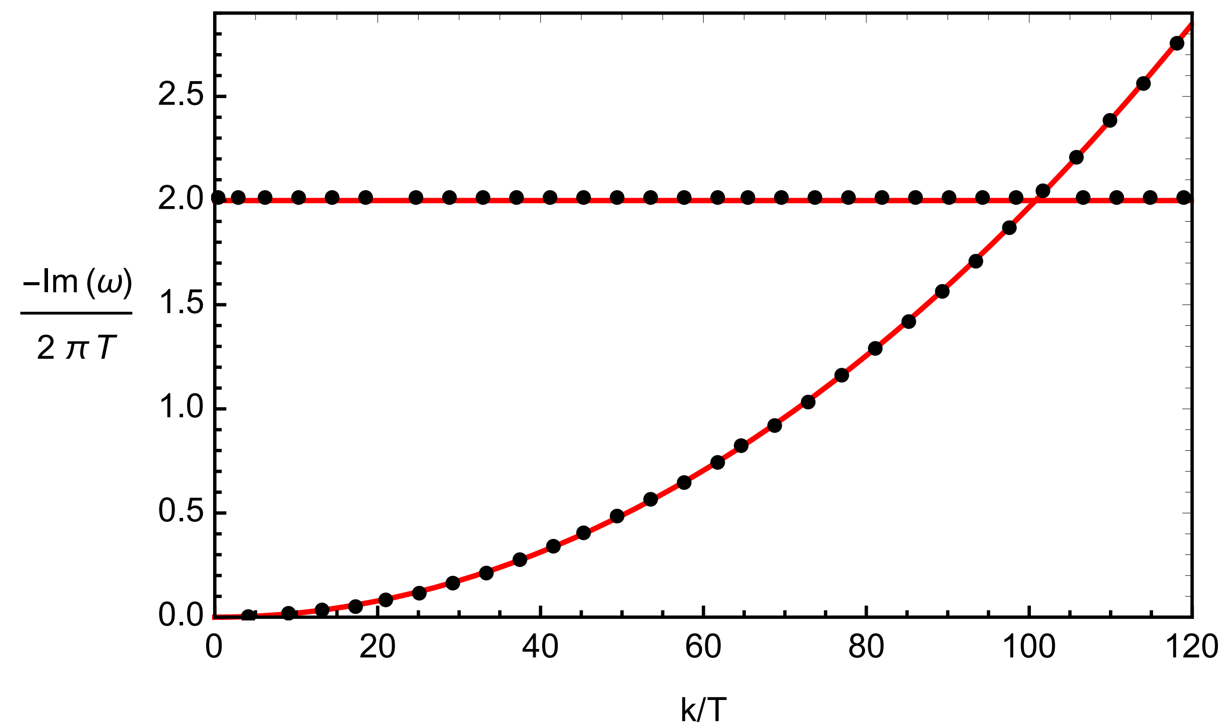
assuming all corrections to dispersion relations are negligible for $k < k_c$

ACTUAL DISPERSION RELATIONS

- For $k < k_c$, corrections to the dispersion relations are in fact parametrically small in the low T limit.

  e.g. axion theory

  $$T = 10^{-3} k_L$$



- In the low T limit

$$\omega_{eq} \to 2\pi T\Delta \qquad\qquad k_{eq}^2 \to \frac{\omega_{eq}}{D} = \frac{2\pi T\Delta}{D}$$

- Small corrections to the dispersion relation give a small imaginary part to k_c .

BREAKDOWN OF HYDRODYNAMICS: AXION THEORY

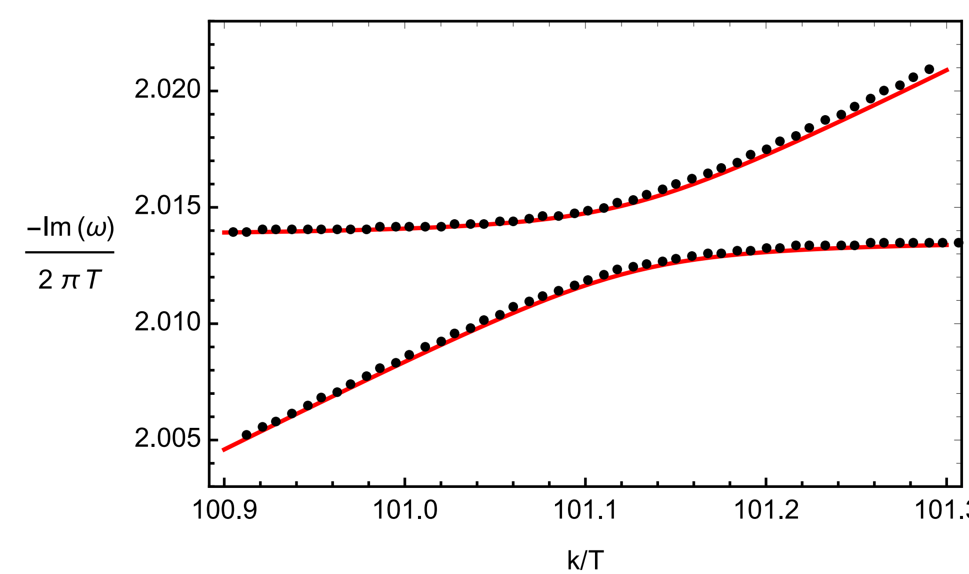
- Can compute $G^{-1}_{\varepsilon\varepsilon}(\omega, k)$ perturbatively close to the expected collision point

$$\omega = -i2\pi T\Delta + \delta\omega \qquad\qquad k^2 = \frac{2\pi T\Delta}{D} + \delta(k^2)$$

- In low T limit $\quad G^{-1}_{\varepsilon\varepsilon}(\omega, k) \propto \left(D\delta(k^2) - i\delta\omega\right)(1 - i\tau\delta\omega) - i\lambda\delta\omega + \ldots$

where $\qquad \tau = \dfrac{9k_L}{32\sqrt{6}\pi^2 T^2} \qquad\qquad \lambda = \dfrac{3\sqrt{3}\pi T}{2k_L}$

$\longrightarrow$ no pole collision at real k



LOCAL EQUILIBRATION SCALES: AXION THEORY

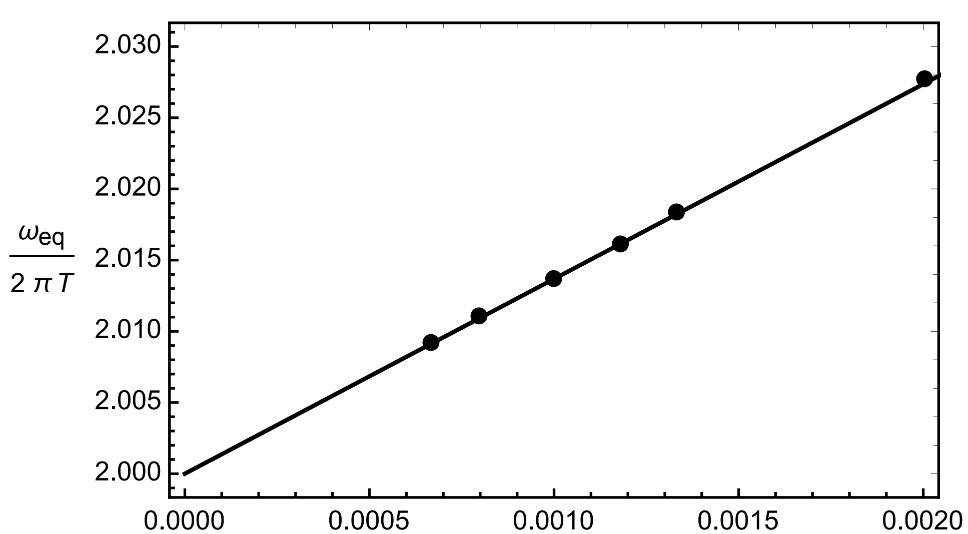
- There is a branch point located at

$$\omega_c = -i4\pi T\left(1 + \frac{8\sqrt{6}\pi T}{9k_L} + \ldots \pm i\left(\frac{2^{13/4}}{3^{3/4}}\left(\frac{\pi T}{k_L}\right)^{3/2} + \ldots\right)\right) \qquad k_c^2 = 4\sqrt{\frac{2}{3}}k_L\pi T\left(1 - \frac{4\sqrt{6}\pi T}{9k_L} + \ldots \pm i\left(\frac{2^{17/4}}{3^{3/4}}\left(\frac{\pi T}{k_L}\right)^{3/2} + \ldots\right)\right)$$

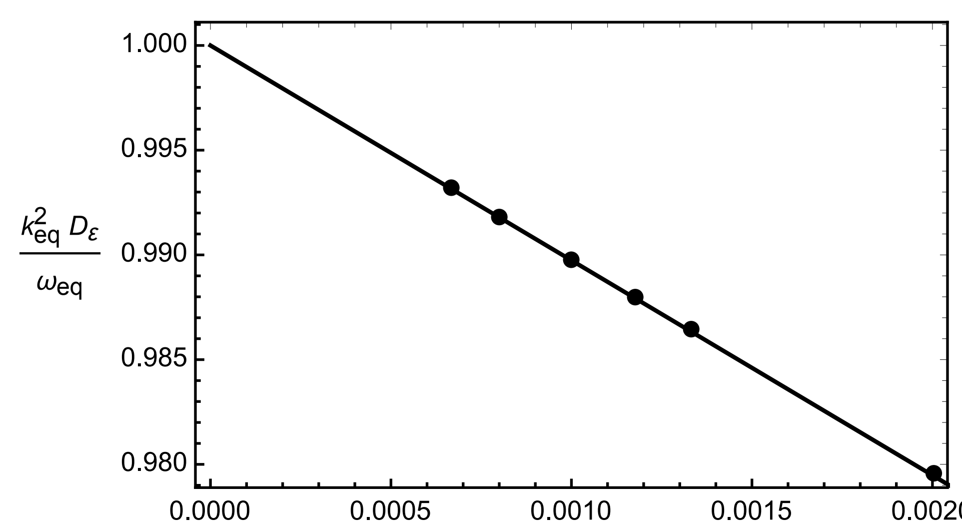
$\longrightarrow$ local equilibration scales

$$\omega_{eq} = 2\pi T\Delta\left(1 + \frac{8\sqrt{6}\pi T}{9k_L} + \ldots\right) \qquad k_{eq}^2 = \frac{\omega_{eq}}{D}\left(1 - \frac{4\sqrt{6}\pi T}{3k_L} + \ldots\right)$$

- Comparison to location of pole collision computed numerically:



T/k_L

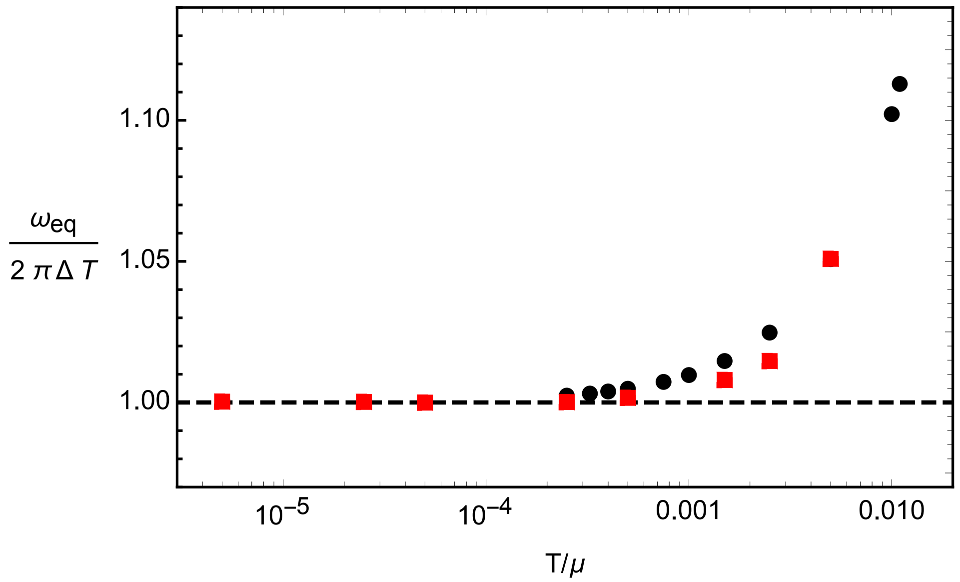


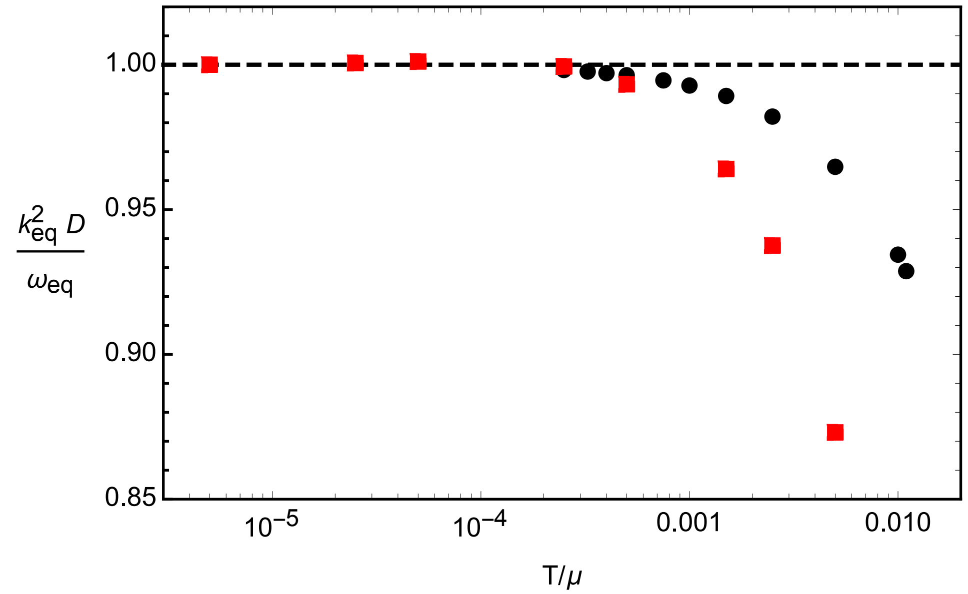
T/k_L

LOCAL EQUILIBRATION SCALES: RN-ADS$_4$

- Everything qualitatively the same for both modes of RN-AdS$_4$

consistent with Withers (1803.08058), Jansen & Pantelidou (2007.14418), Abassi & Tahery (2007.10024)





● diffusion of temperature perturbations

■ diffusion of momentum perturbations

$$\omega_{eq} \to 2\pi T\Delta \qquad k_{eq}^2 \to \frac{\omega_{eq}}{D} = \frac{2\pi T\Delta}{D}$$

SYK CHAIN MODEL

- SYK model is a theory of N interacting fermions.

  At large N and strong coupling: low energy effective action is same as gravity in (nearly)-AdS_2.

  Maldacena, Stanford, Yang (1606.01847), …

- SYK chain model is a generalisation of this with spatial locality ("AdS_2xR")

$$H = i^{q/2} \sum_{x=0}^{M-1} \left(\sum_{\substack{1 \le i_1 < \ldots < i_q \le N \\ ,}} J_{i_1 \ldots i_q, x} \chi_{i_1,x} \cdots \chi_{i_q,x} + \sum_{\substack{1 \le i_1 < \ldots < i_{q/2} \le N \\ 1 \le j_1 < \ldots < j_{q/2} \le N}} J'_{i_1 \ldots i_{q/2} j_1 \ldots j_{q/2}, x} \chi_{i_1,x} \cdots \chi_{i_{q/2},x} \chi_{j_1,x+1} \cdots \chi_{j_{q/2},x+1} \right),$$

  There is one hydrodynamic mode: diffusion of energy.

  At strong coupling, may expect this to look like state with AdS_2xR in the IR.

  Gu, Stanford, Qi (1609.07832), …

SYK CHAIN MODEL

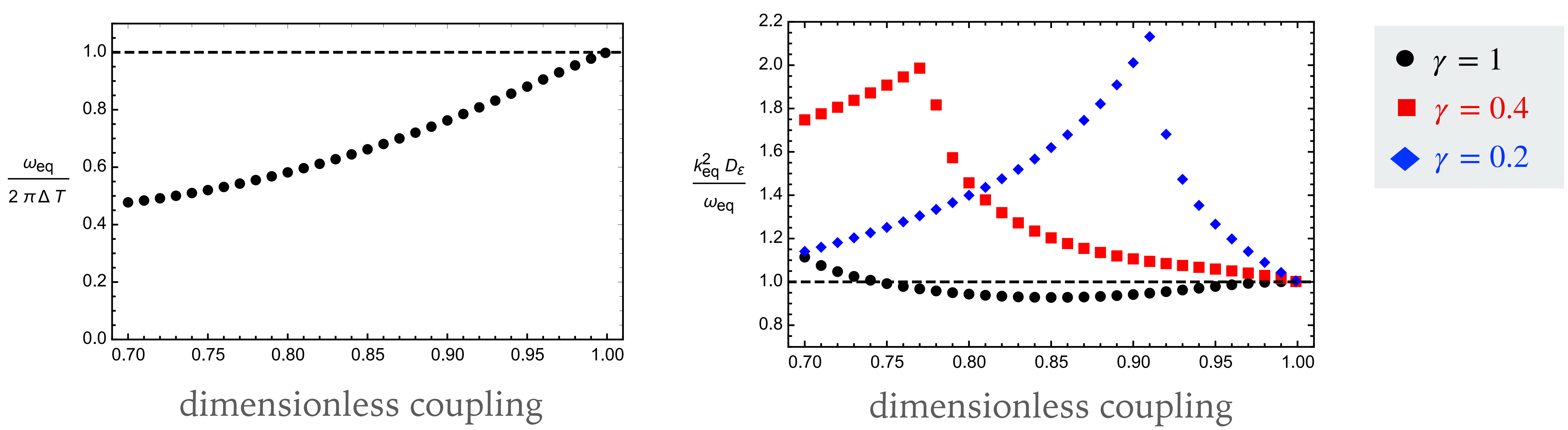
- $G_{\varepsilon\varepsilon}(\omega, k)$ can be calculated exactly in the limit $N \gg q^2 \gg 1$. Choi et al (2010.08558)

  At strong coupling

$$G_{\varepsilon\varepsilon}(\omega, k) \propto \frac{\Gamma\left(\frac{1}{2} - h(k)\right)\Gamma\left(h(k) - \frac{i\omega}{2\pi T}\right)}{\Gamma\left(\frac{1}{2} + h(k)\right)\Gamma\left(1 - h(k) - \frac{i\omega}{2\pi T}\right)}$$

- These are the IR modes of AdS_2 with $\Delta(k) = h = \frac{1}{2}\left(1 + \sqrt{9 + 4\gamma\left(\cos(k) - 1\right)}\right)$.

  The breakdown of hydrodynamics is as in the holographic theories:



SUMMARY

- Looked at (certain) states governed by an AdS_2 fixed point in the IR.

  Simple relations between equilibration timescales and low energy properties

$$\omega_{eq} \to 2\pi T\Delta \qquad\qquad k_{eq}^2 \to \frac{\omega_{eq}}{D} = \frac{2\pi T\Delta}{D}$$

- These simple relations are a consequence of two properties

  ✳ Lifetime of longest-lived non-hydrodynamic mode set by Δ .

  ✳ Corrections to the quadratic approximation to the hydrodynamic dispersion relation are parametrically small for $k < k_c$.

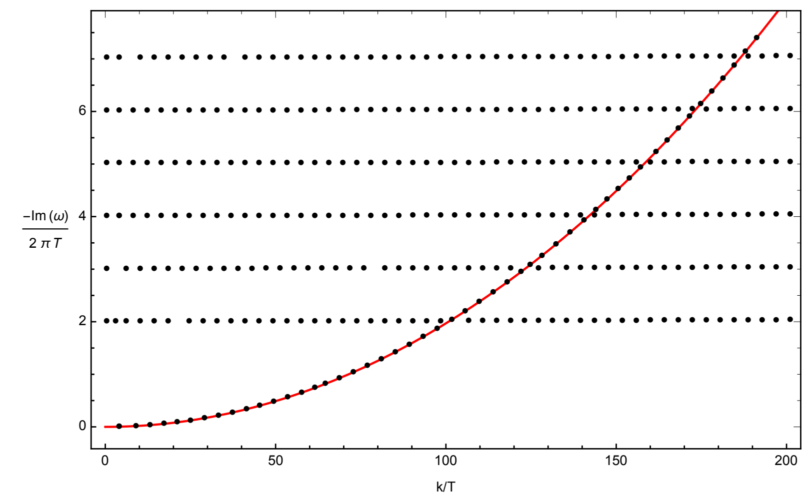
OPEN QUESTIONS

- Generalisations:

  ∗ other AdS_2 fixed points (reason to be confident, at least in some cases)
  ∗ AdS_2 with non-universal deformation
  ∗ non-AdS_2 fixed points (e.g. Lifshitz)
  ∗ multiple diffusion modes in one Green's function (e.g. complex SYK chain)

- The "breakdown" of hydrodynamics:

see also Moitra, Sake, Trivedi (2005.00016)



- Saturation of the bound $D \lesssim \omega_{eq} k_{eq}^{-2}$ that follows from assumption that ω_{eq}/k_{eq} is set by an underlying effective light cone speed.

Hartman, Hartnoll, Mahajan (1706.00019)

THANK YOU!