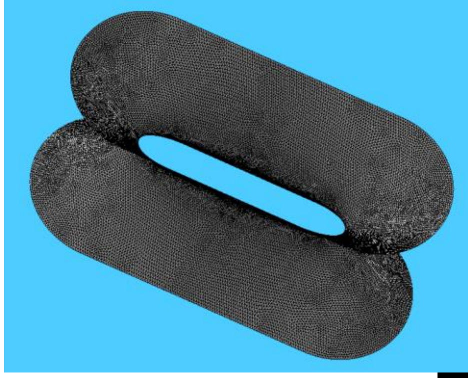
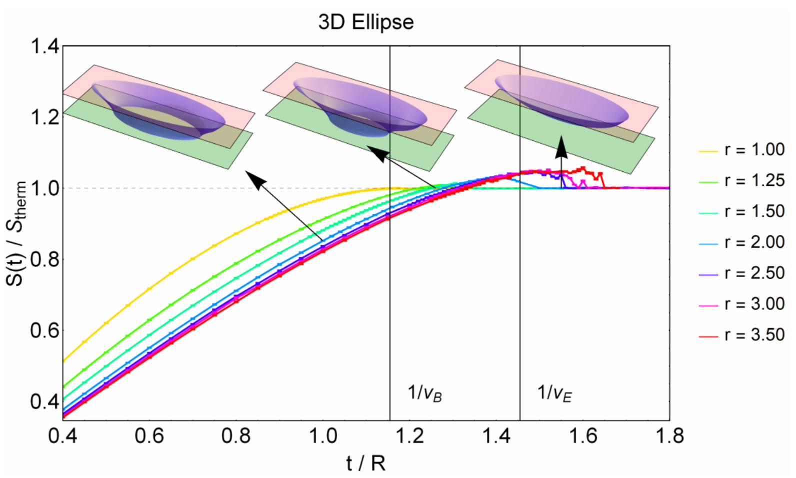


ENTANGLEMENT ENTROPY AT STRONG COUPLING – THERMALIZATION AND EXPANDING SPACETIMES

INCLUDING A SHORT INTRO ON MEMBRANE THEORY

With Márk Mezei and Jorge Casalderrey-Solana, Christian Ecker and David Mateos

Reference: 2001.03172 (PRL), 2011.08194



Wilke van der Schee

University of Ljubljana
Ljubljana, 17 December 2020

OUTLINE

A quench and entanglement saturation

- Entanglement tsunami: entanglement and butterfly velocities
- Membrane theory: a hydrodynamic approach

Numerics: chaotic thermalisation in stadia and ellipsoids

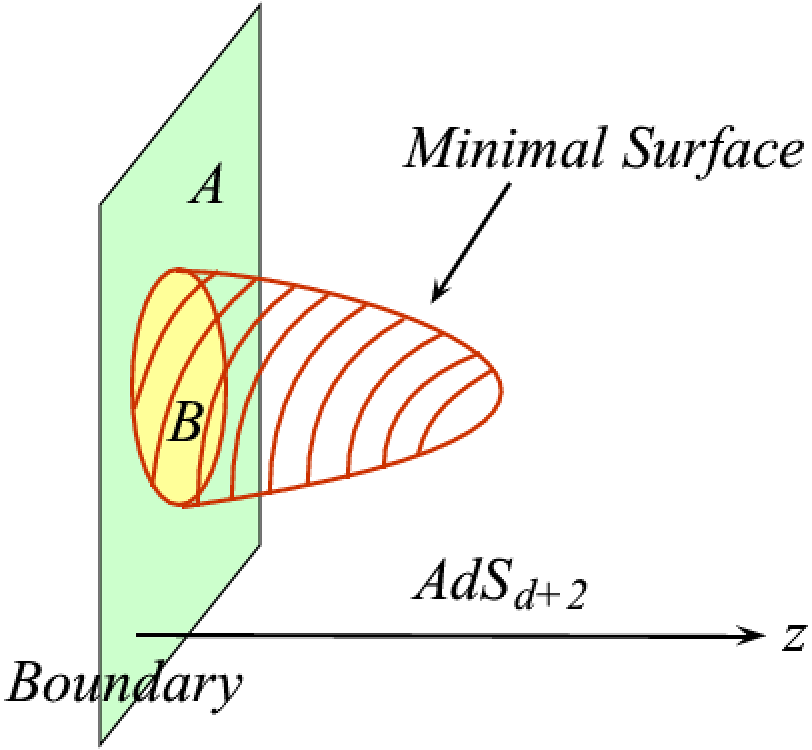
- Black holes thermalize fast for a large class of shapes

Entanglement in de Sitter space-time

- From boundary cosmological horizon to
  bulk event, apparent and entanglement horizons

HOLOGRAPHIC ENTANGLEMENT ENTROPY

Entanglement entropy CFT ≅ geometry AdS



Prescription:

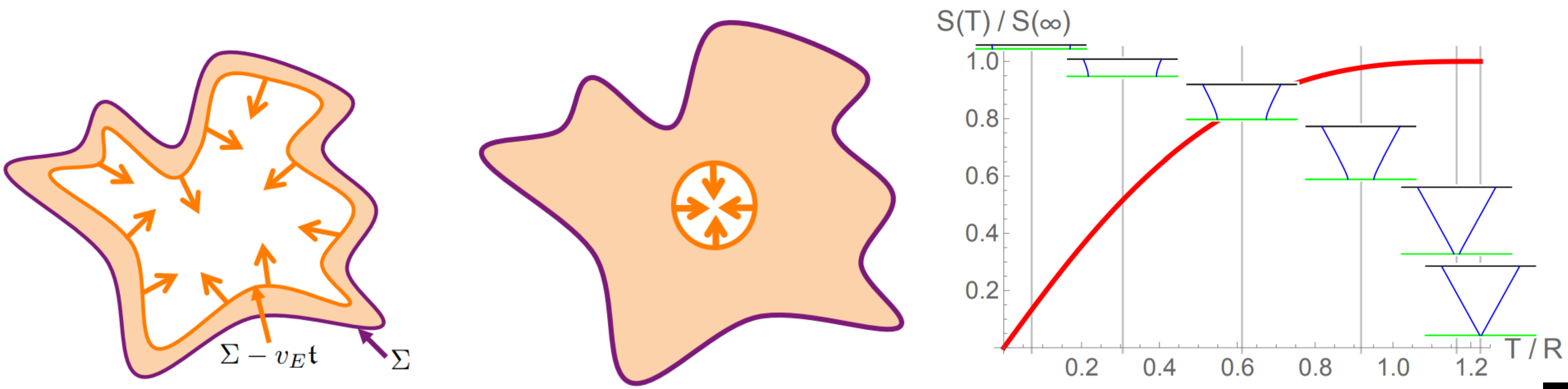
- Find minimal surface in AdS that matches entangling region on boundary
- AdS is ‘smaller’ at large z (IR), so minimal surface goes to IR for large regions
- Time dependence: minimal → extremal (harder)

Tatsuma Nishioka, Shinsei Ryu and Tadashi Takayanagi, Holographic Entanglement Entropy: An Overview (2009)

VAIDYA QUENCH: ENTANGLEMENT TSUNAMI

Entanglement grows after quench to thermal value

- Early time (t~1/T) ignored: details of quench are unimportant (Vaidya)
- Stripes: constant growth (v_E=1 for d=2)



Early time linear growth: $v_E = \sqrt{-\frac{a(z)}{z^{2(d-1)}}}\Big|_{\max} = \frac{\left(\frac{d-2}{d}\right)^{(d-2)/(2d)}}{\left(\frac{2(d-1)}{d}\right)^{(d-1)/d}}$

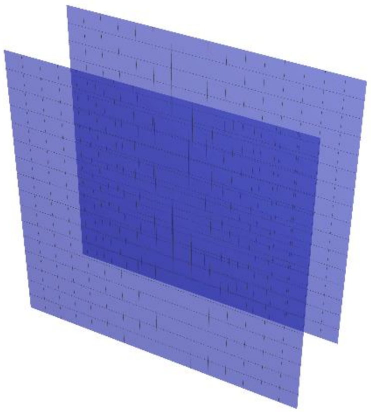
Hong Liu and S. Josephine Suh, Entanglement Tsunami: Universal Scaling in Holographic Thermalization (2013)

TWO EXAMPLES

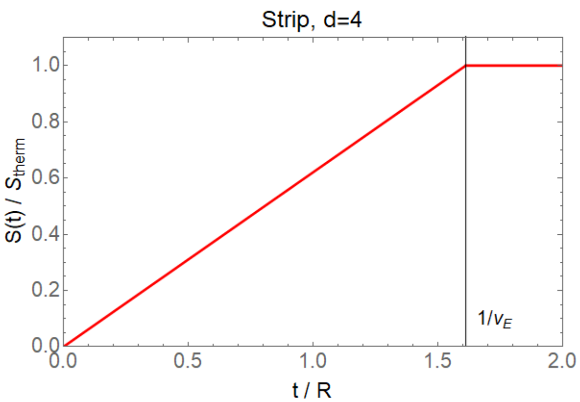
$$\eta = 2(d-1)/d$$

Stripe

- Linear growth till (discontinuous) saturation

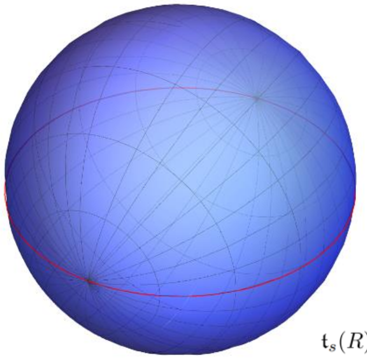


$$v_E = \frac{(\eta-1)^{(\eta-1)/2}}{\eta^{\eta/2}}$$



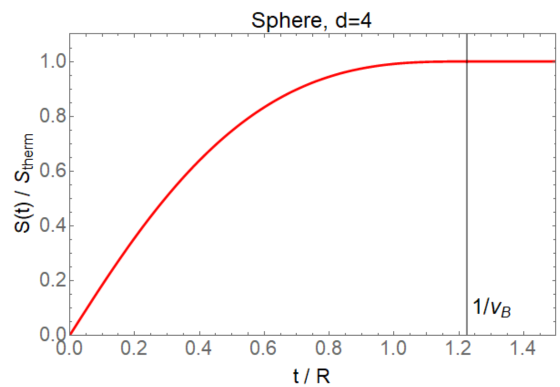
Sphere

- Shape dependent growth till (critical) saturation



$$t_s(R) = \frac{1}{c_E}R - \frac{d-2}{4\pi T}\log R + O(R^0)$$

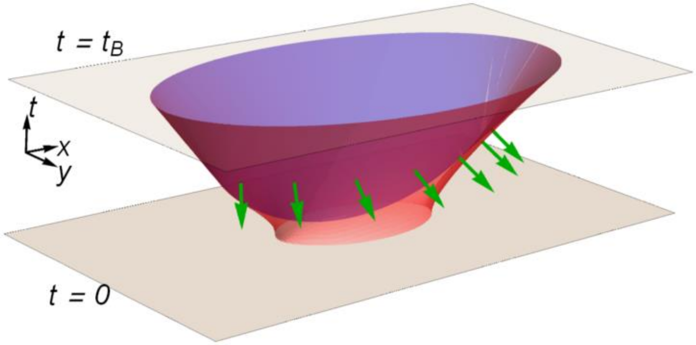
$$v_B = 1/\sqrt{\eta}$$



Hong Liu and S. Josephine Suh, Entanglement Tsunami: Universal Scaling in Holographic Thermalization (2013)

MEMBRANE THEORY: HOLOGRAPHY

Metric (t>0): $$ds^2 = \frac{1}{z^2}\left[-a(z)dt^2 - \frac{2}{b(z)}\,dtdz + d\vec{x}^2\right]$$

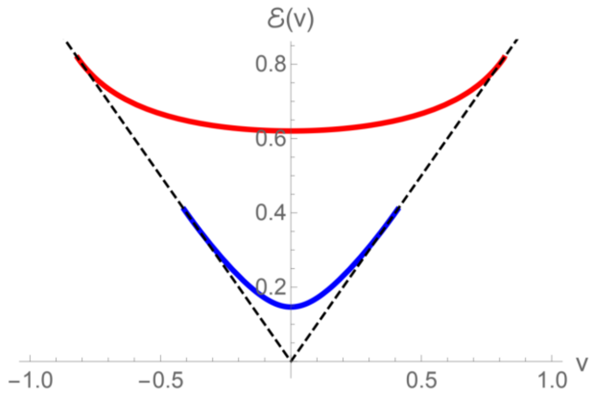


Important limit: times and regions much larger than 1/T: possible to integrate out holographic direction

Extremal surface:

$$S = s_{\text{th}} R^{d-1} \int d\tau d\Omega\ \rho^{d-2} \sqrt{1 + \frac{(\partial_\Omega \rho)^2}{\rho^2}}\, \mathcal{E}(v)$$
$$= s_{\text{th}} R^{d-1} \int \text{darea}\ \frac{\mathcal{E}(v)}{\sqrt{1-v^2}},$$

$$\mathcal{E}(v) \equiv \sqrt{\frac{-a'(\zeta)}{2(d-1)\zeta^{2d-3}}}\Bigg|_{\zeta = c^{-1}(v^2)}$$



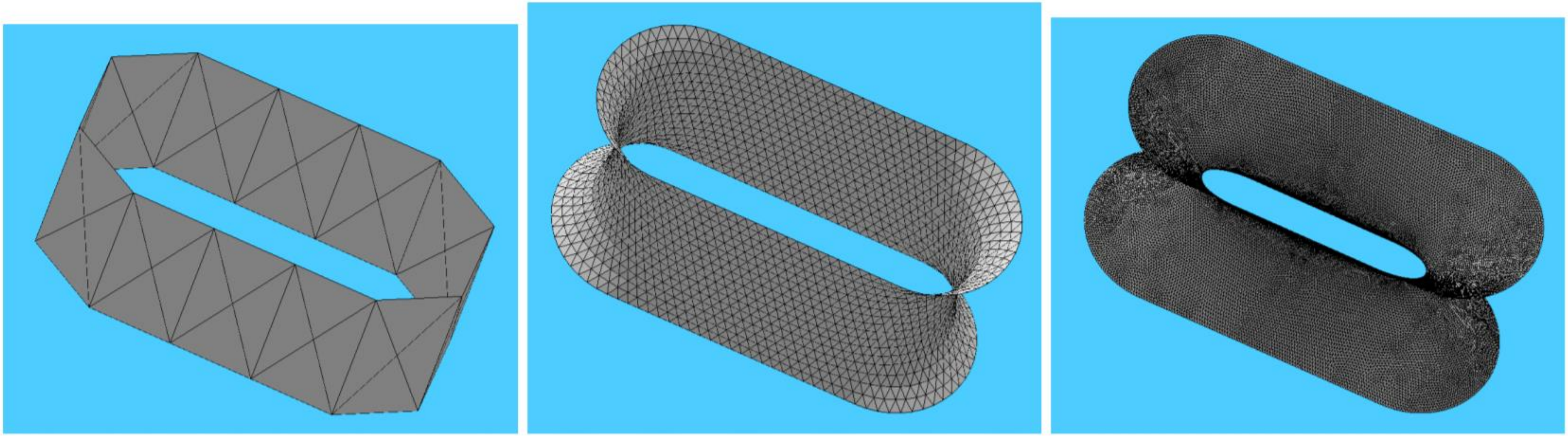
Tension for neutral (red) and charged (blue) black brane

Important: extremising → minimising

Mark Mezei , Membrane theory of entanglement dynamics from holography (2018)

NUMERICS: SURFACE EVOLVER

Refine and relax surface:

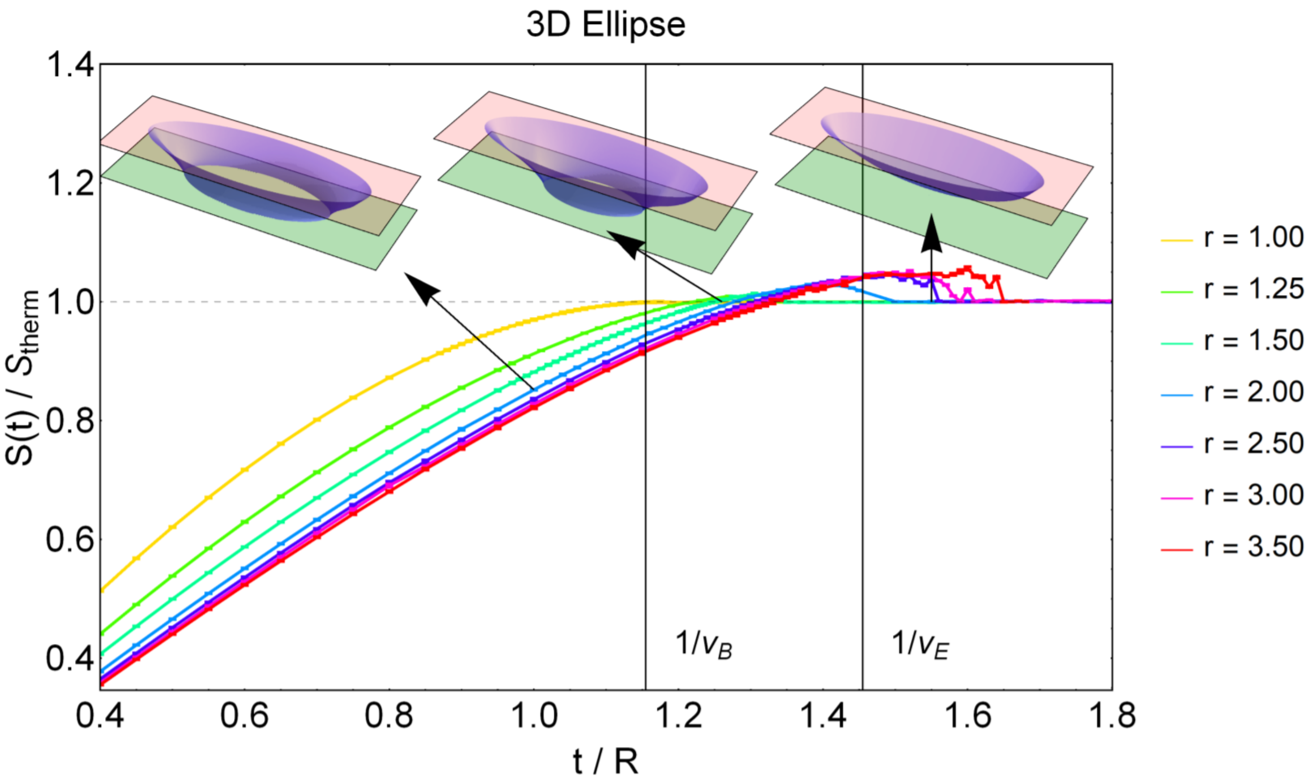


Relax by minimising membrane action:

```
gVu  := {V; u; g 25;}
gogo := {
  scale:=0.001;
  r 2; gVu 50;
  scale:=0.0025;
  r; g 12000;
  V; u; g 29000;
  gVu 8000;
  scale:=0.0005;
  g 30000;
  r; gVu 15;
  g 20000;
  r; gg  10;
  g 100000
},
```

3D RESULTS

Thermalises *almost* at butterfly speed for spheres; otherwise slower:



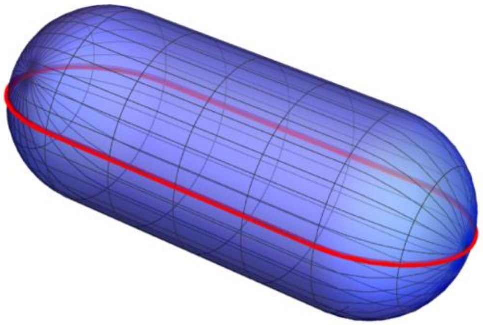
Not as interesting, as no fast thermalisation bound conjectured

4D RESULTS

Two options with same amount of symmetry:

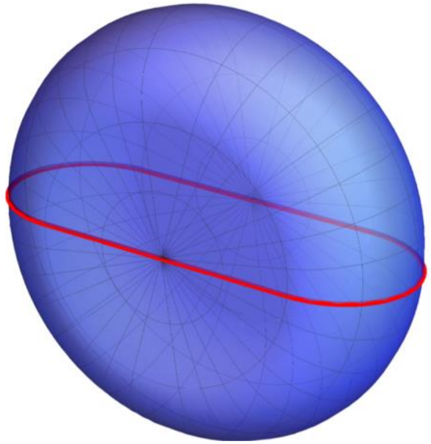
Tube

- Infinite ratio: cylinder
  - Saturates at butterfly speed
- Ratio > 1

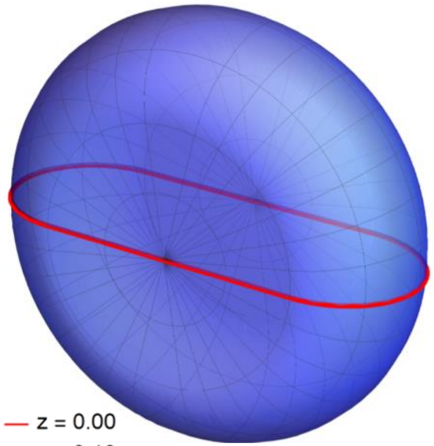


Icecream waffle

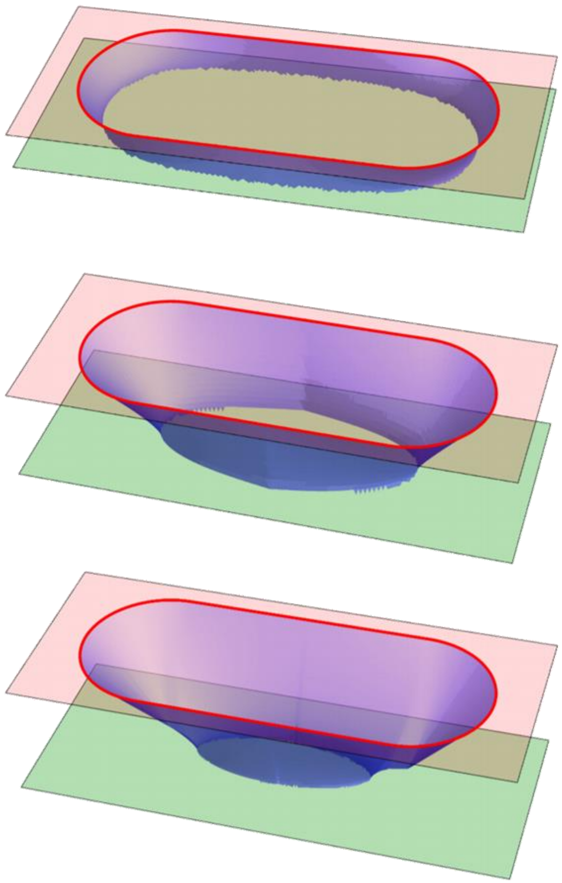
- Infinite ratio: stripe
  - Saturates at entanglement speed
- Ratio < 1

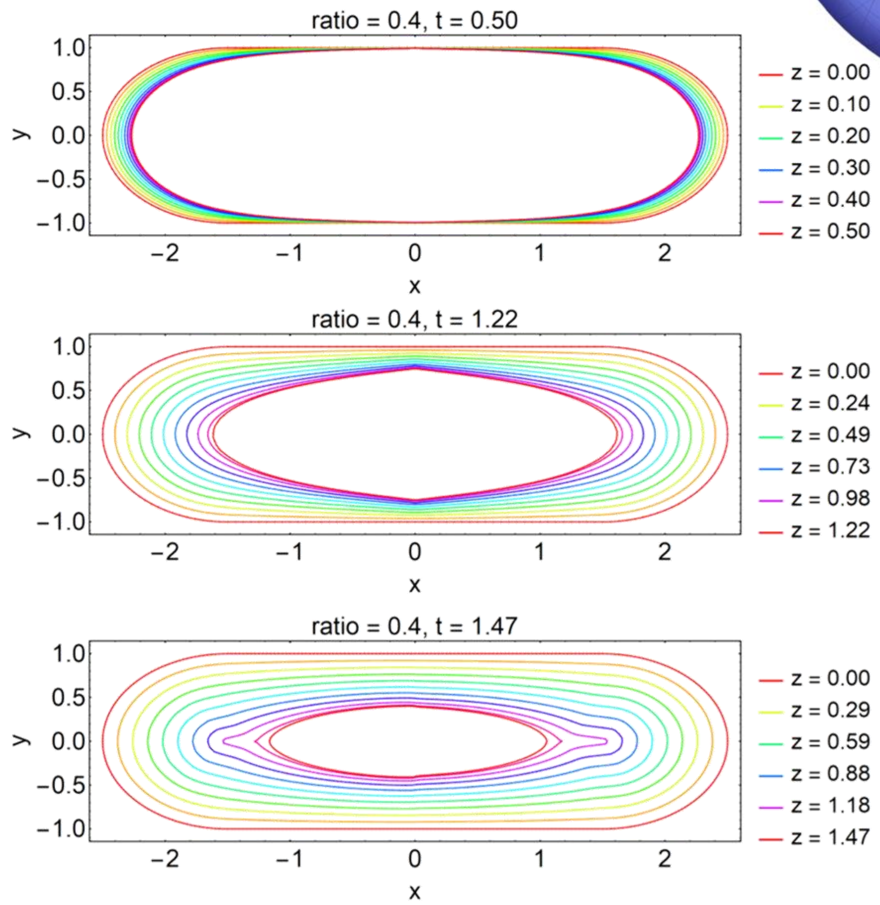


4D WAFFLE

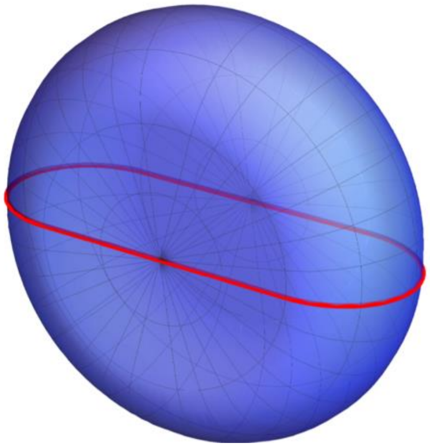


Membrane at several times; also: cusp formation

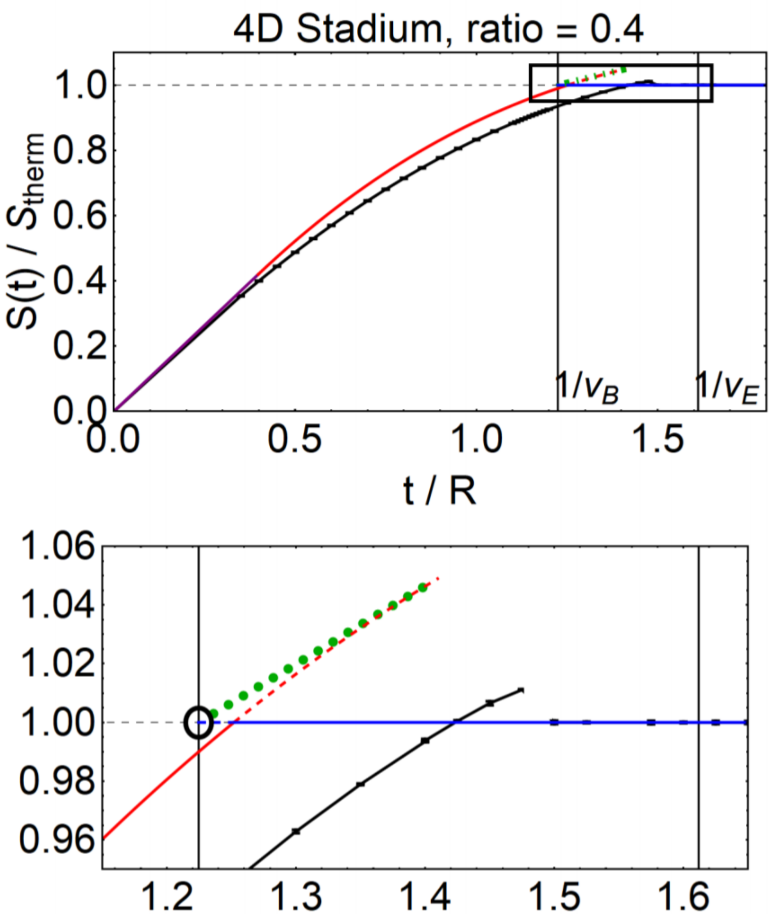




4D WAFFLE



Thermalisation curve:

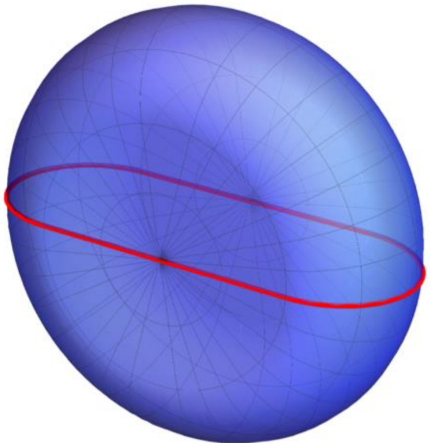


Analytic bound (red):

- Construct membrane with constant butterfly velocity + from t=t': v=0 part:

$$S_{\max}[A(t)] = s_{\rm th} \min_{0\leq t'\leq \min(t,t_B)} \left[(\mathrm{vol}\,(A) - \mathrm{vol}\,(A')) + v_E\, \mathrm{area}\,(A')\; t' \right]$$

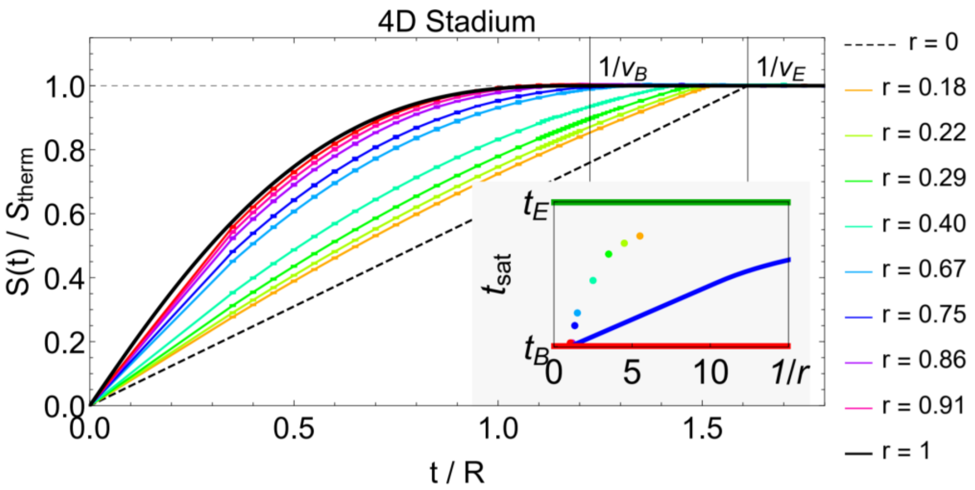
- Corollary: butterfly time is saturation time iff $t' = t_B$



4D WAFFLE

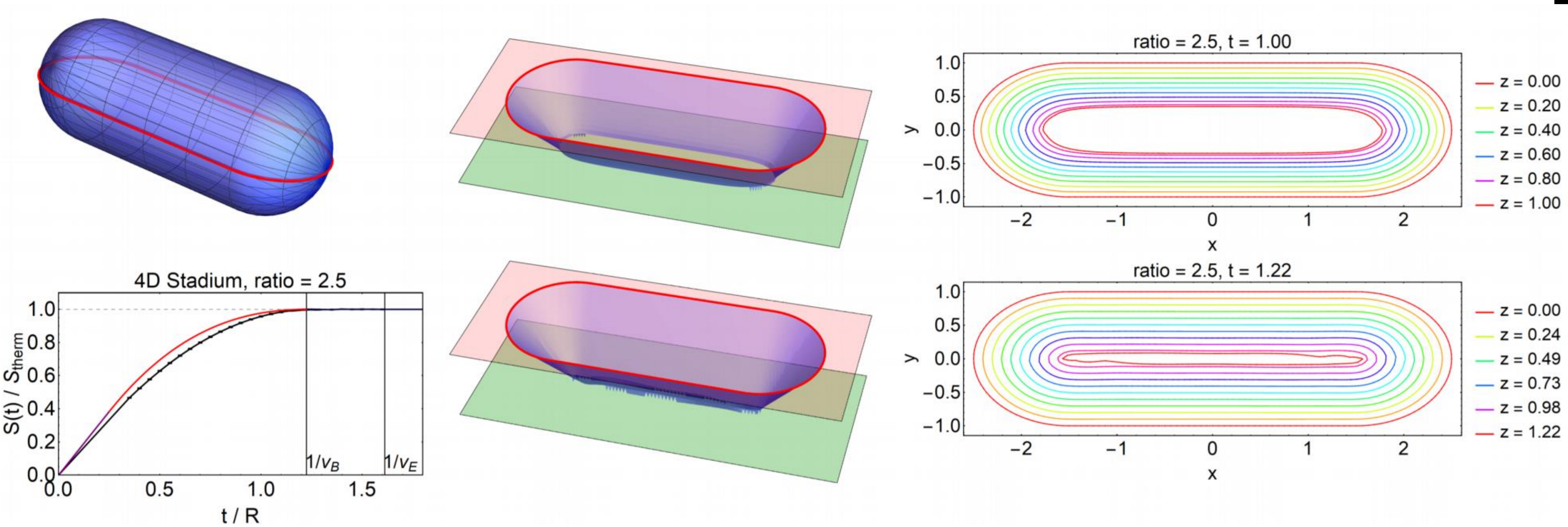
Saturation times for many ratios:

- Saturates approximately at t_B for $r > 0.75$
- Indeed approaches t_E (stripe) for large ratios

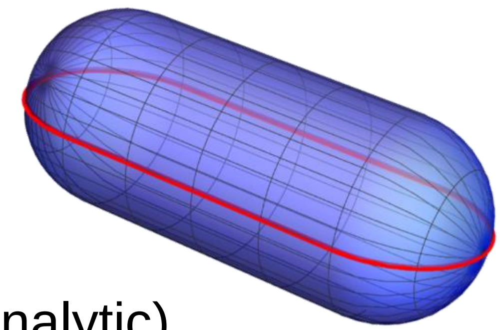


4D TUBE

Membrane at several times; also: cusp formation

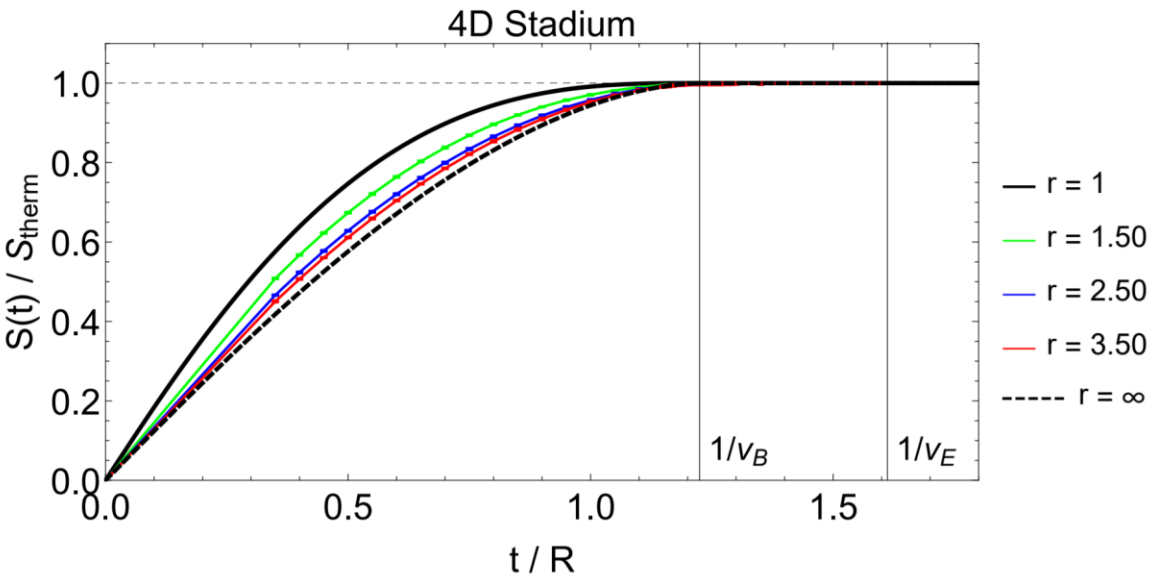


4D TUBE



Saturation times for many ratios:

- Nicely interpolates between sphere and cylinder (analytic)
- *All of them saturate at butterfly time*
  - Shows genericity of black holes as fast scramblers



IMPORTANT LESSON

Saturation of the butterfly bound on entanglement saturation:

- Analytically known: cylinders saturate, stripes do not
- New result: true for general `class' around spheres and stripes:
  - Waffles don't saturate (from r <~ 0.75)
  - Tubes always saturate (making bound generic)

Difference between cylinders and stripes:

- Stripes have `two large directions', i.e. much larger than R
  - R is radius largest inscribed sphere
- Somewhat intuitive: thermalisation over `large' space takes longer

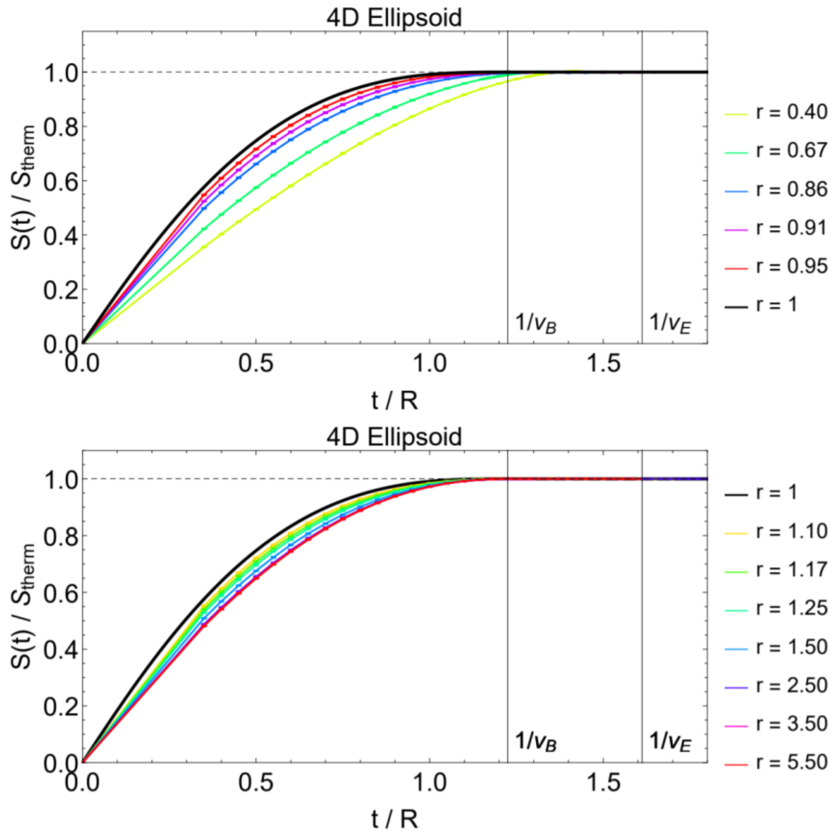
Independent: charged spheres do not always saturate

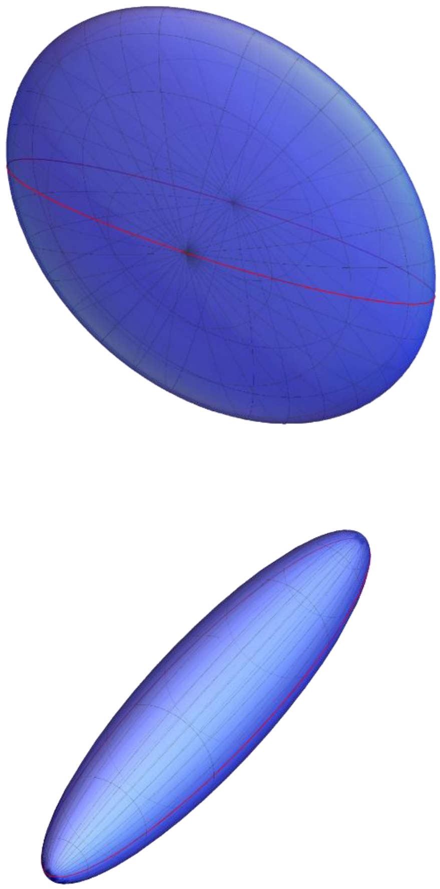
- Schwarzschild black holes really the fastest scramblers?
- Comparison with free theory? (typically infinite saturation time…)

MORE GENERAL: ELLIPSOIDS

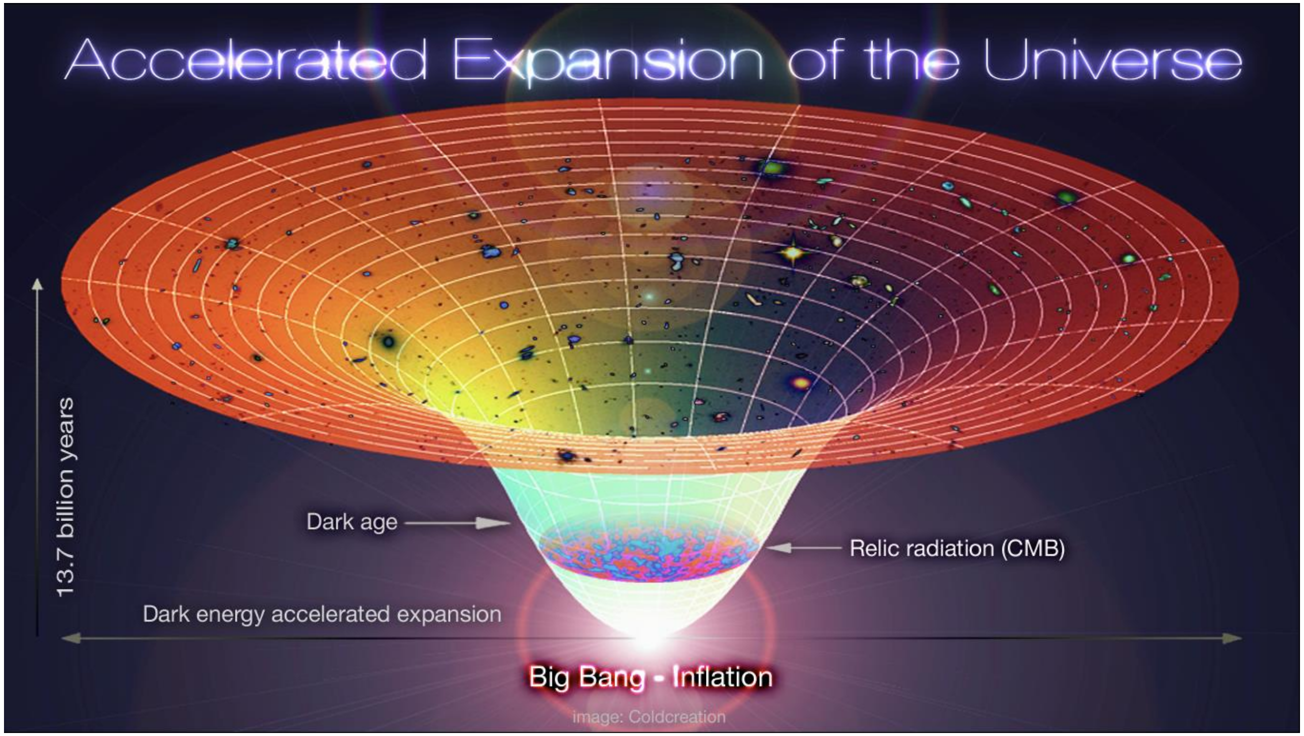
Similar for ellipsoids: `cylinder-like' thermalises in butterfly time

- Non-trivial: no analytic arguments for limiting cases





ENTANGLEMENT ON DE SITTER$_4$



NON-CONFORMAL MODEL ON DE SITTER$_4$

De Sitter is conformally flat: almost trivial for CFT

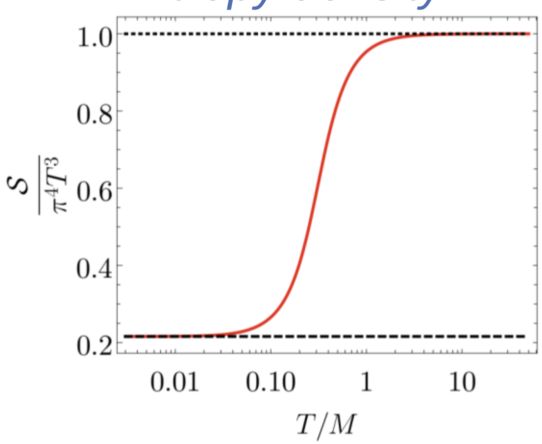
- Break scale invariance by V(Φ) with source M=1:

$$S=\frac{2}{8\pi G}\int_{\mathcal{M}} d^5x\sqrt{-g}\left(\frac{1}{4}R[g]-\frac{1}{2}(\partial\phi)^2-V(\phi)\right)$$

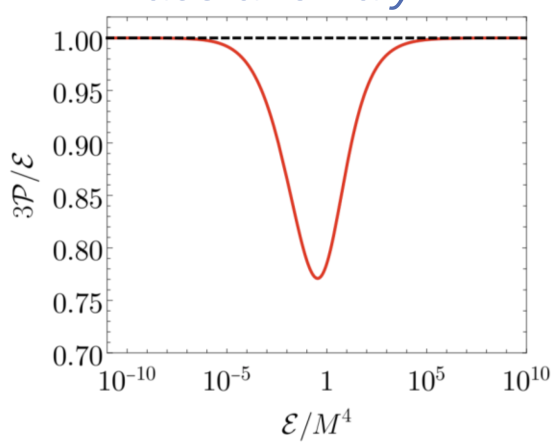
$$L^2V(\phi)=-3-\frac{3}{2}\phi^2-\frac{1}{3}\phi^4+\left(\frac{1}{3\phi_M^2}+\frac{1}{2\phi_M^4}\right)\phi^6-\frac{1}{12\phi_M^4}\phi^8$$

- Leads to non-trivial EOS and bulk viscosity (no shear considered):

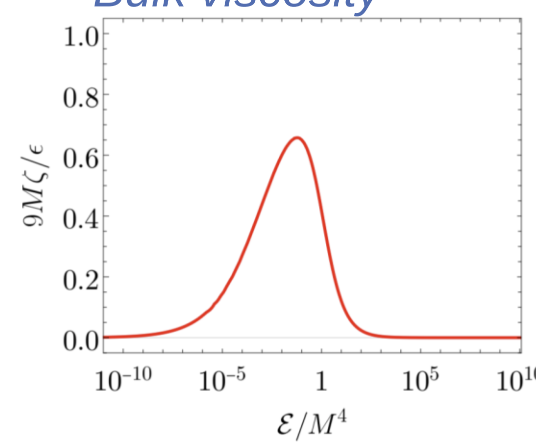
Entropy density



Trace anomaly



Bulk viscosity



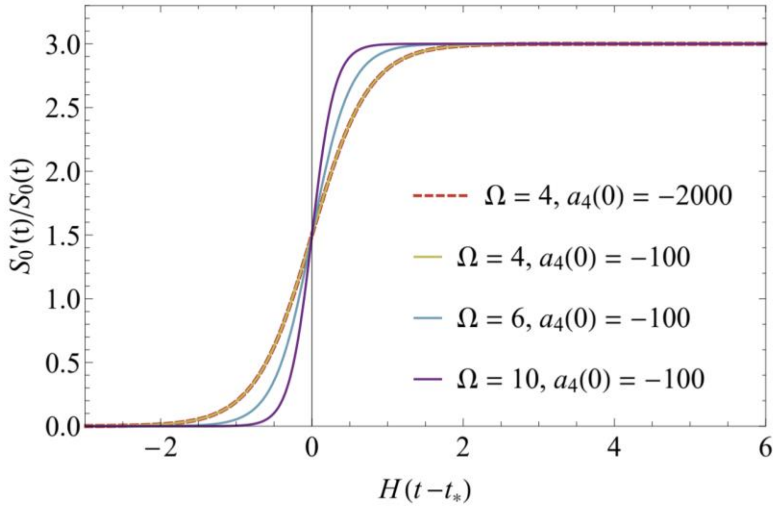
HOW WE SET UP A STATE

Non-trivial boundary metric: $ds_{\rm b}^2 = -{\rm d}t^2 + S_0(t)^2 {\rm d}\vec{x}^2 \,. \quad S_0(t) = e^{Ht}$

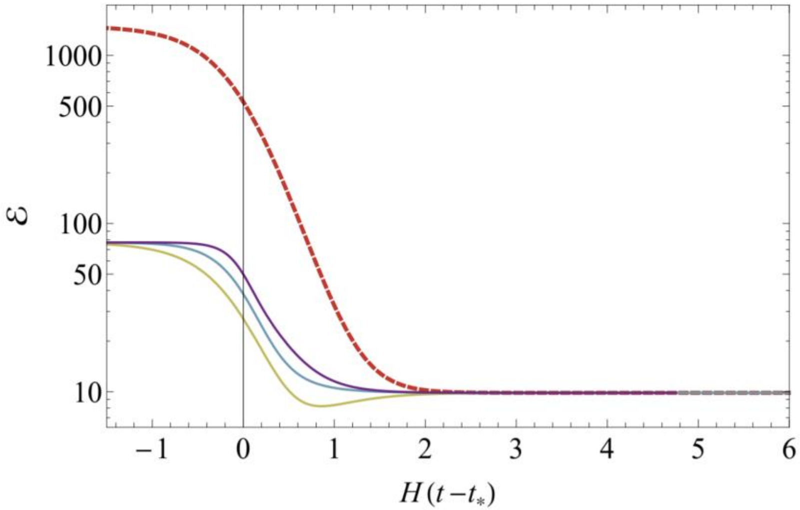
Start with thermal (high-temperature) state in flat space

- Quench system by suitable fast tanh to constant Hubble parameter
- Energy density decreases towards final `vacuum energy' (VE)
- Final (Bunch-Davis)-VE is ambiguous → chose scheme with zero VE

Hubble parameter



Energy density

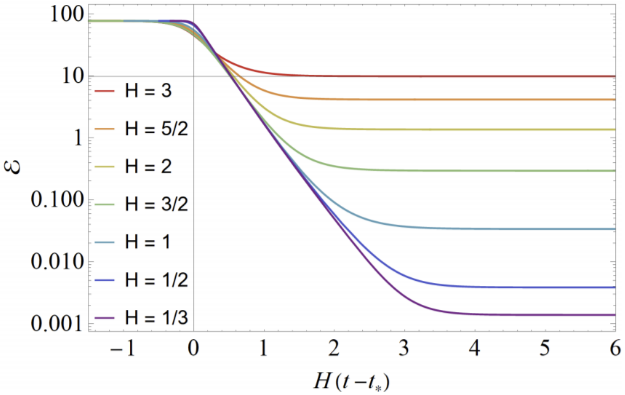


TIME EVOLUTION OF THE PROTOCOL

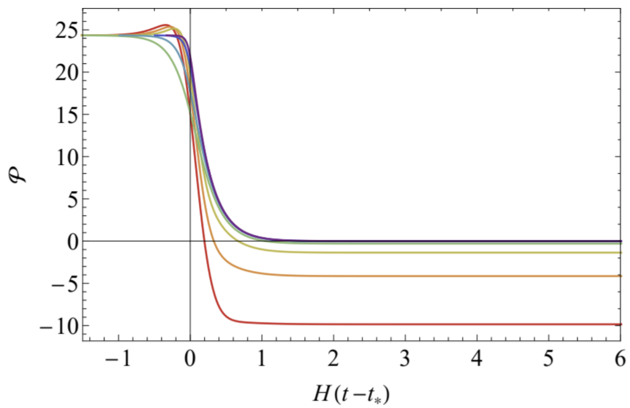
Evolution of stress tensor for different Hubble constants

- Energy density decreases towards VE (can be renoralised to zero)
- Pressure decreases, changes sign and becomes –VE
- Enthalpy is scheme independent, decays due to expansion

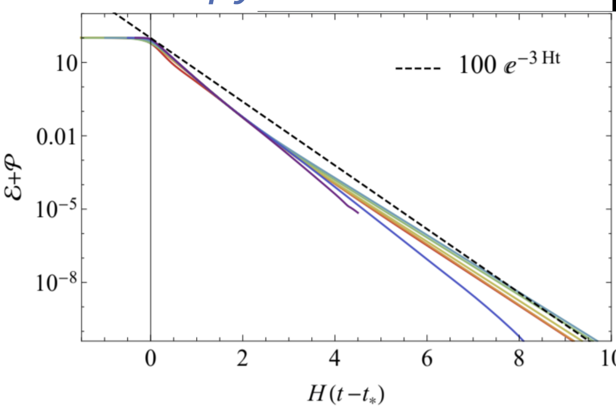
Energy density



Pressure



Enthalpy



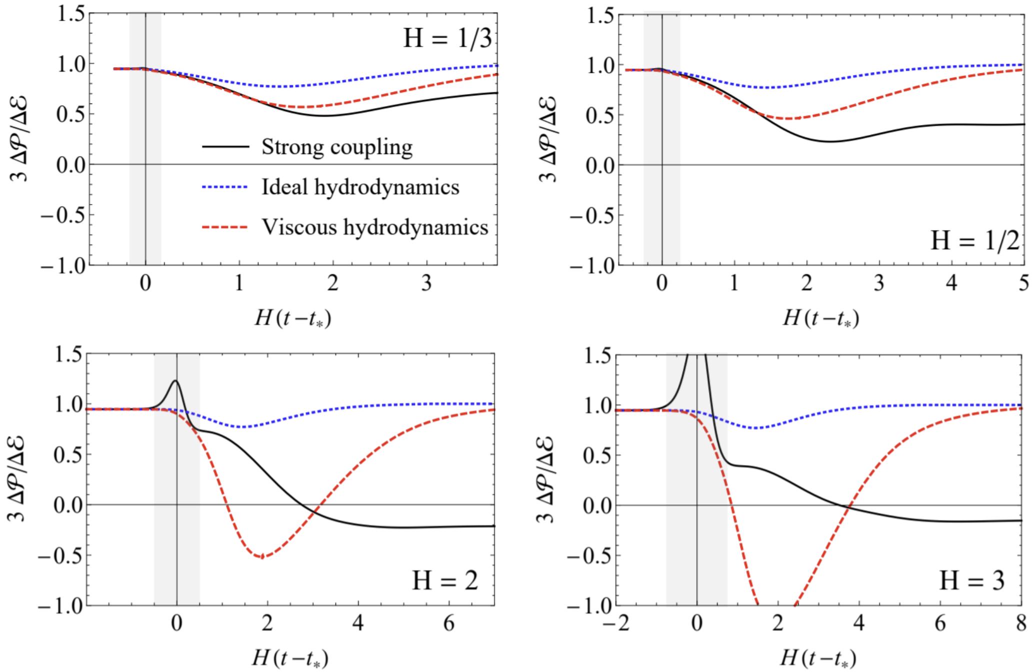
THE APPROACH TOWARDS HYDRODYNAMICS

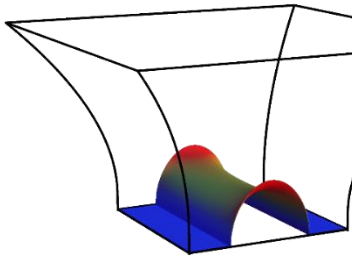
$$T_\perp^{\mu\nu} = P(\varepsilon)\Delta^{\mu\nu} - \eta(\varepsilon)\sigma^{\mu\nu} - \zeta(\varepsilon)\Delta^{\mu\nu}(\nabla\cdot u)\,,$$
$$\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$$

Non-trivial hydrodynamic prediction

$$\Delta\mathcal{P}^{\rm hydro}(t) \equiv \Delta p_{\rm eq}(\Delta\mathcal{E}(t)) - 3H\zeta(\Delta\mathcal{E}(t)) + O(H^2)\,,$$

- All done with `subtracted' quantities (e.g. VE = 0)
- A puzzle: at which energy to take bulk viscosity? (irrelevant at 1st order)
- Viscous hydro works for small H (gradients). Negative `EOS' for large H.



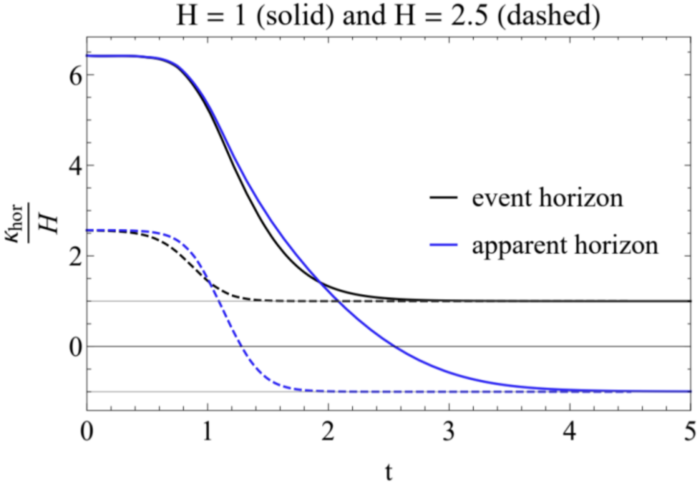


BLACK HOLE THERMODYNAMICS

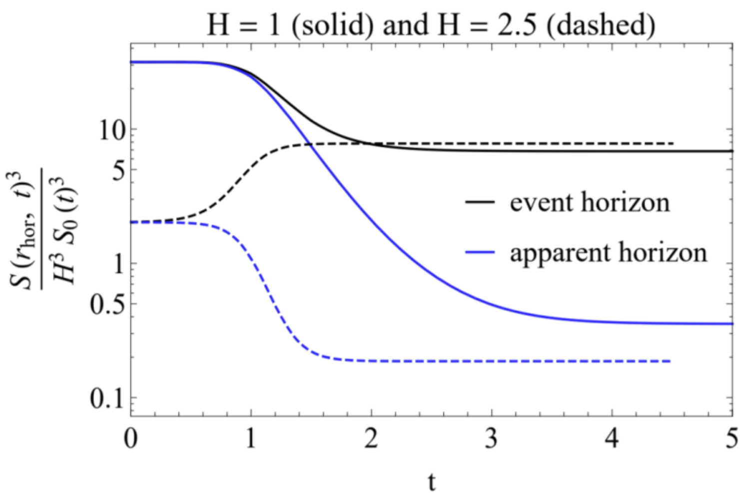
Keep track of bulk event and apparent horizons (EF coordinates)

- Dynamical setting: horizons not coincide at late times:
- Surface gravities can be shown analytically: $\kappa_{\mathrm{EH}} = -\kappa_{\mathrm{AH}} = H$
  EH confirms Hawking's temperature in de Sitter
- Area density apparent horizon vanishes for conformal theory

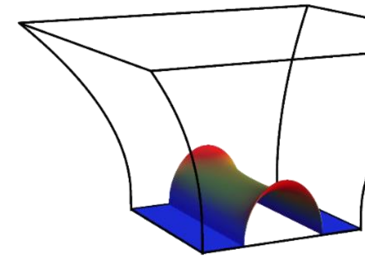
Surface gravity



Area densities



Willy Fischler, Sandipan Kundu and Juan Pedraza, Entanglement and out-of-equilibrium dynamics in holographic models of de Sitter QFTs (2013)
Alex Buchel, Entanglement entropy of N = 2* de Sitter vacuum (2019)



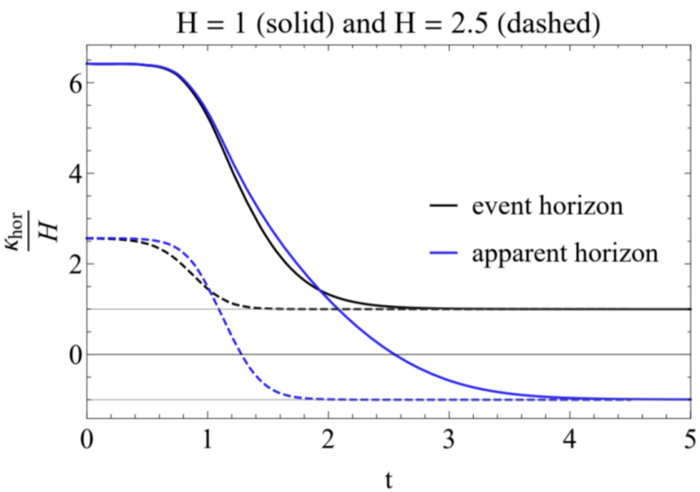
BLACK HOLE THERMODYNAMICS

Several interpretational issues

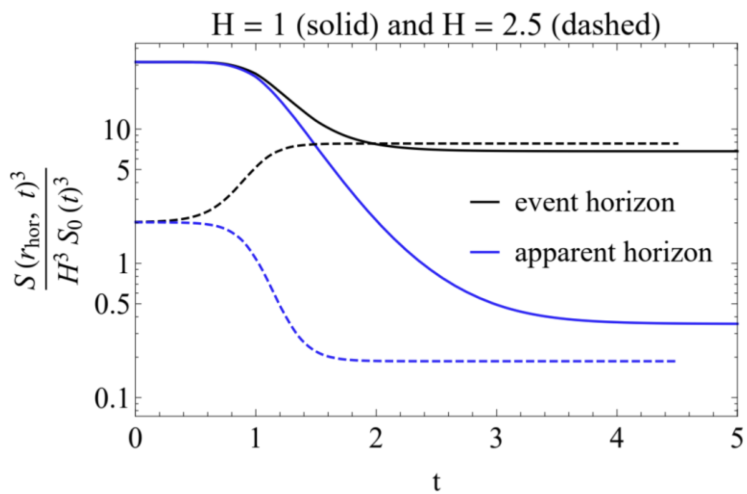
- Expanding space: mapping boundary to bulk horizon not clear
- Apparent horizon: time slicing dependent
- In general: no volume law entropy density expected

Resolution → entanglement entropy is well defined

Surface gravity



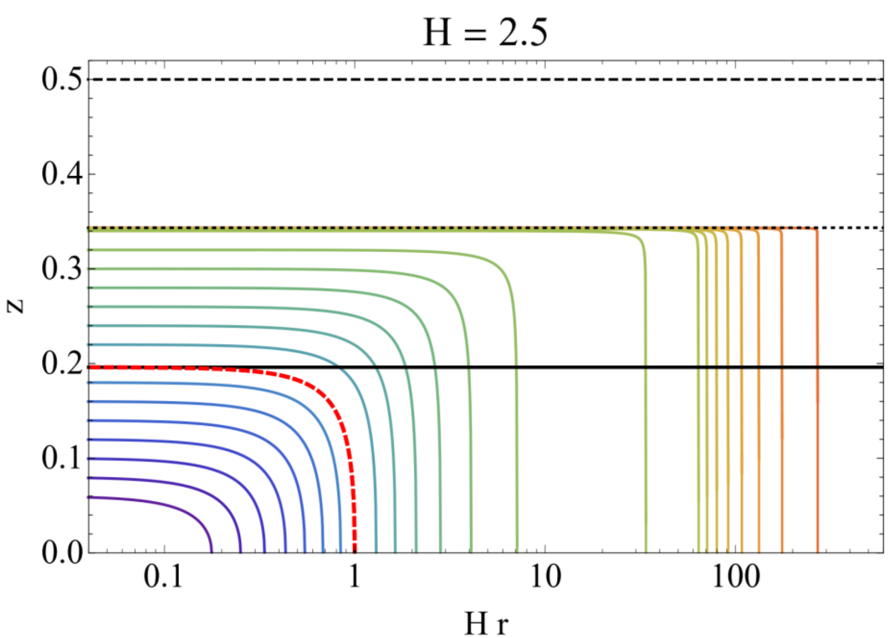
Area densities



ENTANGLEMENT IN DE SITTER

Extremal surfaces dual to spherical entangling regions:

- Large entangling regions probe beyond event horizon
- A new `entanglement horizon' forms, between AH and EH, with zero surface gravity

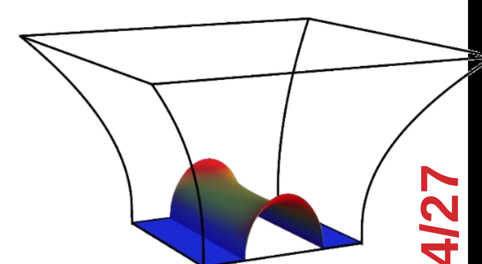


Cosmological horizon

Apparent horizon: $T = -\frac{H}{2\pi}$

Entanglement horizon: $T = 0$

Event horizon: $T = \frac{H}{2\pi}$

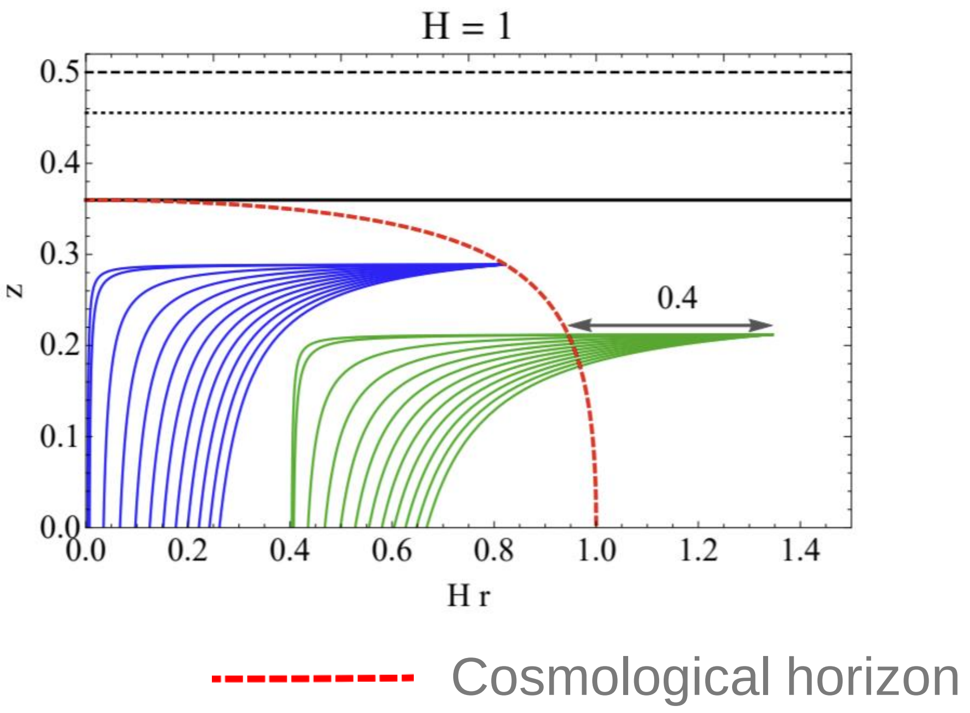


Jorge Casalderrey, Christian Ecker, David Mateos and WS, Strong-coupling dynamics and entanglement in de Sitter space (2020)

ENTANGLEMENT IN DE SITTER

Extremal surface dual to cosmological horizon:

- Separates points in the bulk from which light can reach the (boundary) origin
- *Boundary cosmological horizon → full bulk cosmological horizon*

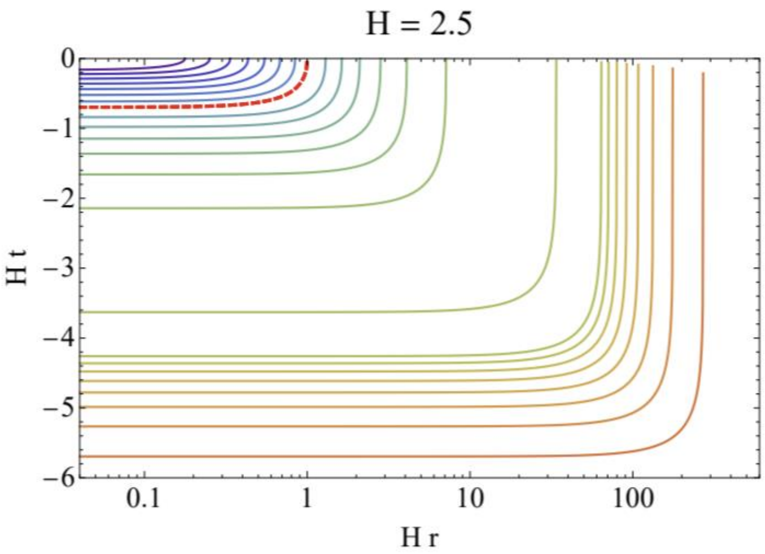


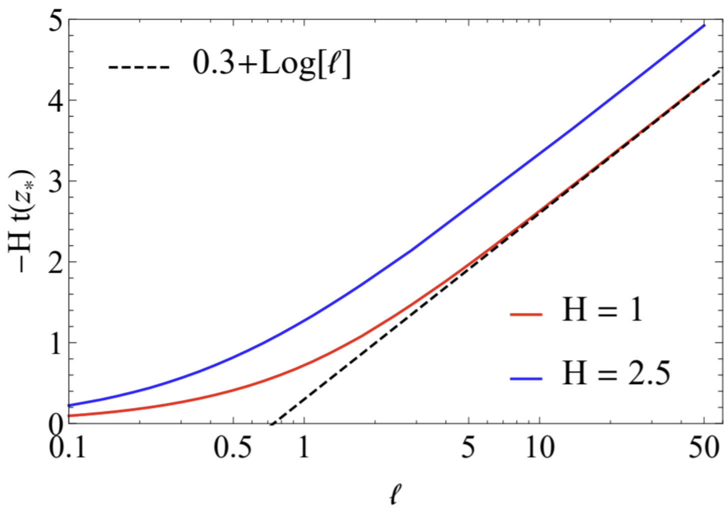
Jorge Casalderrey, Christian Ecker, David Mateos and WS, Strong-coupling dynamics and entanglement in de Sitter space (2020)

ENTANGLEMENT IN DE SITTER

Extremal surface go backward in time

- Time at the deepest point grows exactly as log(l) for large l
- *Implies that `entanglement horizon' contribution has a constant instead of volume law contribution*
- Standard `area law' divergence still applies





Cosmological horizon

DISCUSSION

Entanglement entropy thermalisation

- Membrane theory: large time and length behaviour (hydro):
  - Extremisation → minimisation problem

Surface evolver

- Solved variety of shapes: waffles, tubes, ellipsoids, 3D + 4D …
- Butterfly thermalisation *for large class of shapes*
- Mechanism quite clear by analytic bound:
  - Excludes measure zero for black holes as fast scramblers

Entanglement in de Sitter

- Event and apparent horizon differ: negative temperature AH (?)
- Extremal surfaces beyond cosmological horizon probe behind EH
- Extremal surface cosmological horizon extends into bulk as bulk CH
- Area law divergence + constant term from `entanglement horizon'

ENTANGLEMENT IN 1+1 CFT, VAIDYA

Vaidya: far-from-equilibrium quench, instantaneous & homogeneous

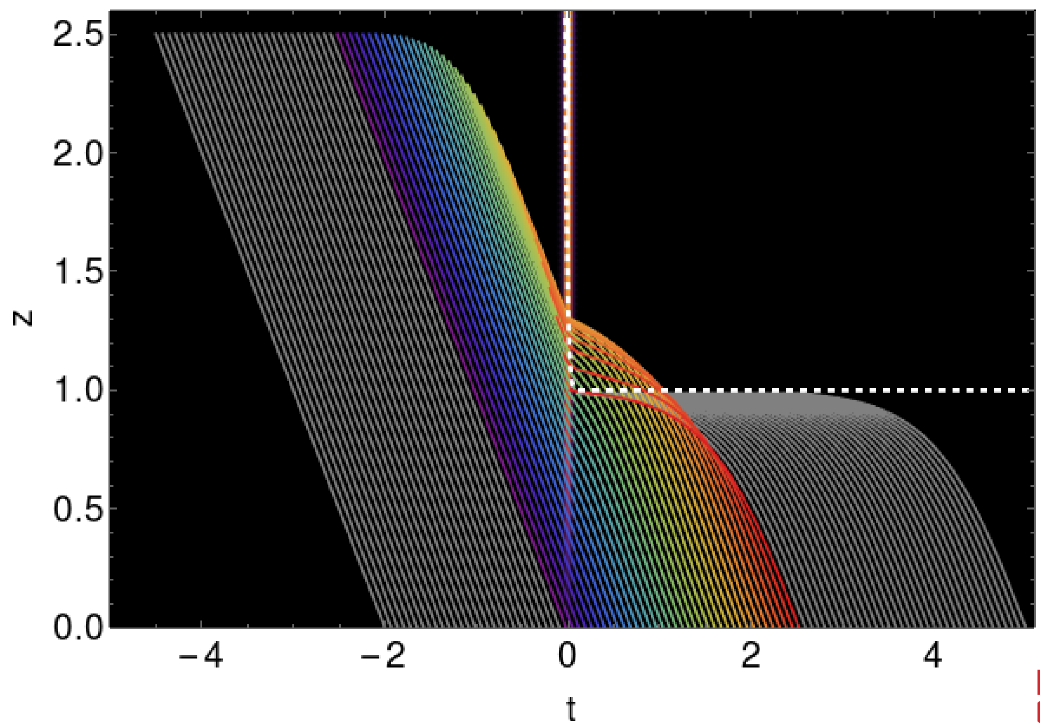
$$ds^2 = \frac{1}{z^2}(-(1 - M(t)z^2)dt^2 - 2dtdz + dx^2)$$

- Interesting: coloured geodesics/surfaces cross matter shell
- Some geodesics can cross horizon, but never at x=0

Time-AdS projection (no x)

- **Size region: 5**
- Vaidya matter shell at t = 0
- Horizon: dashed line
- Coloured extremal surfaces cross shell (till L/2)

NB: Eddington-Finkelstein coordinates



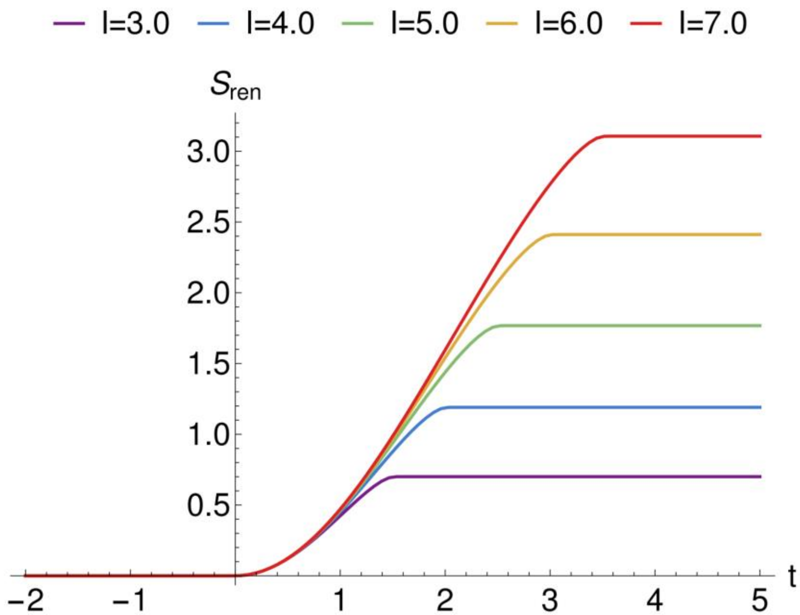
Javier Abajo-Arrastia, João Aparício and Esperanza López, Holographic evolution of entanglement entropy (2010)
C. Ecker, D. Grumiller, M.M. Sheikh-Jabbari, WS and P. Stanzer, Quantum Null Energy Condition and its (non)saturation in 2d CFTs (2019)

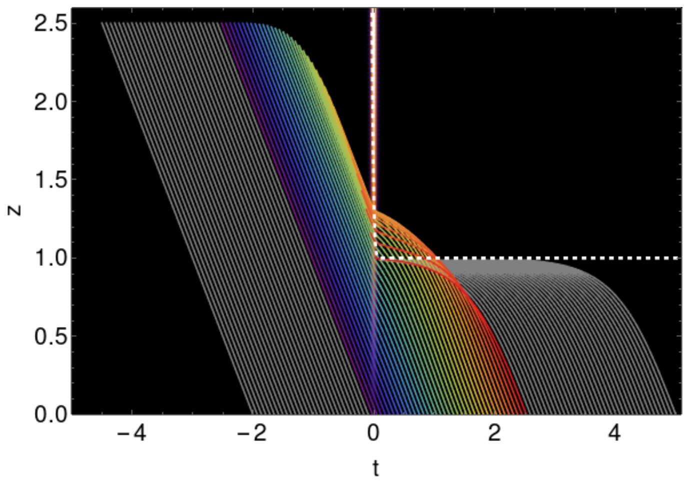
ENTANGLEMENT IN 1+1 CFT, VAIDYA

Vaidya: far-from-equilibrium quench, instantaneous & homogeneous

$$ds^2 = \frac{1}{z^2}(-(1 - M(t)z^2)dt^2 - 2dtdz + dx^2)$$

- Time evolution, and saturation when geodesics do not cross shell anymore (pure thermal)





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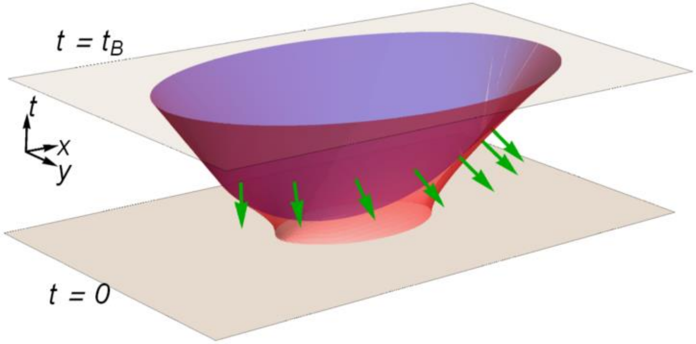
MEMBRANE THEORY: HOLOGRAPHY



Metric (t>0): $ds^2 = \frac{1}{z^2}\left[-a(z)dt^2 - \frac{2}{b(z)}dtdz + d\vec{x}^2\right]$

Rescaling: $t \equiv R\tau\,, \quad r(t,\Omega) \equiv R\,\rho(\tau,\Omega)\,, \quad z(t,\Omega) \equiv \zeta(\tau,\Omega)$

Extremal surface:

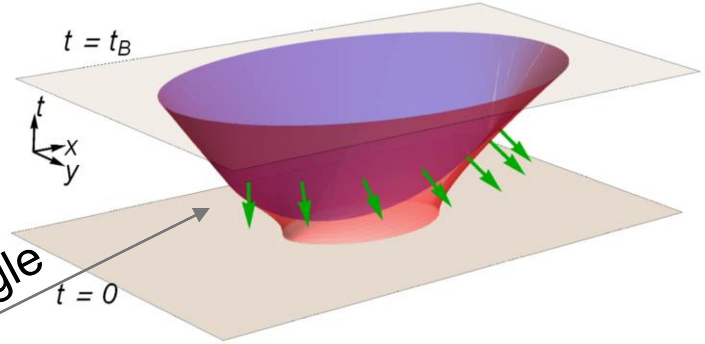
$$S = s_{\rm th} R^{d-1} \int d\tau d\Omega\, \frac{\rho^{d-2}}{\zeta^{d-1}} \sqrt{Q}\,,$$

$$Q \equiv (\partial_\tau \rho)^2 - a(\zeta)\left(1 + \frac{(\partial_\Omega \rho)^2}{\rho^2}\right)$$

Important: holographic ζ direction has no derivative (large R limit)

Mark Mezei , Membrane theory of entanglement dynamics from holography (2018)

MEMBRANE THEORY: HOLOGRAPHY



Extremal surface, several tricks

$$S = s_{\rm th} R^{d-1} \int d\tau d\Omega \, \frac{\rho^{d-2}}{\zeta^{d-1}} \sqrt{Q}\,,$$

$$Q \equiv (\partial_\tau \rho)^2 - a(\zeta) \left(1 + \frac{(\partial_\Omega \rho)^2}{\rho^2}\right)$$

Integrate out ζ: algebraic E-L: $v^2 \equiv \frac{(\partial_\tau \rho)^2}{1+\frac{(\partial_\Omega \rho)^2}{\rho^2}} = a(\zeta) - \frac{\zeta a'(\zeta)}{2(d-1)}$

New action in terms of `membrane':

$$S = s_{\rm th} R^{d-1} \int d\tau d\Omega \, \rho^{d-2} \sqrt{1 + \frac{(\partial_\Omega \rho)^2}{\rho^2}}\, \mathcal{E}(v)$$

$$= s_{\rm th} R^{d-1} \int \text{darea} \, \frac{\mathcal{E}(v)}{\sqrt{1-v^2}}\,,$$

$$\mathcal{E}(v) \equiv \sqrt{\frac{-a'(\zeta)}{2(d-1)\zeta^{2d-3}}}\Bigg|_{\zeta = c^{-1}(v^2)}$$

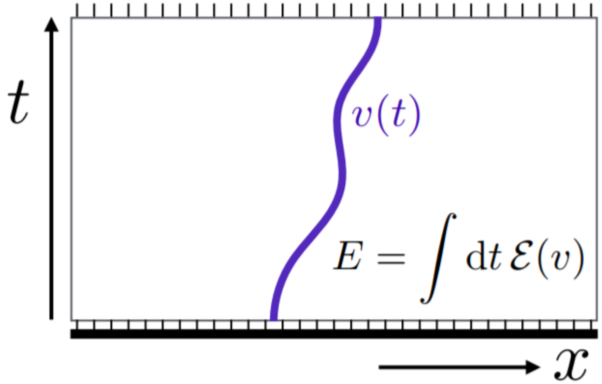
Important: now minimization of timelike membrane in 3+1 Minkowski

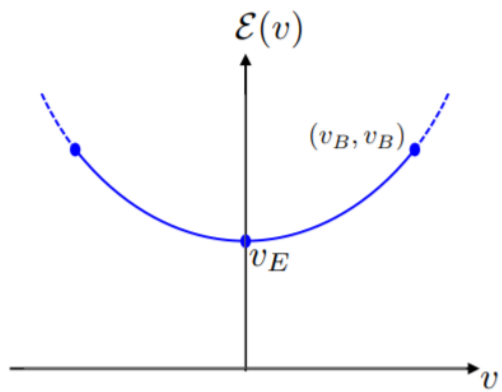
(was: extremisation of spacelike membrane in 4+1 AdS)

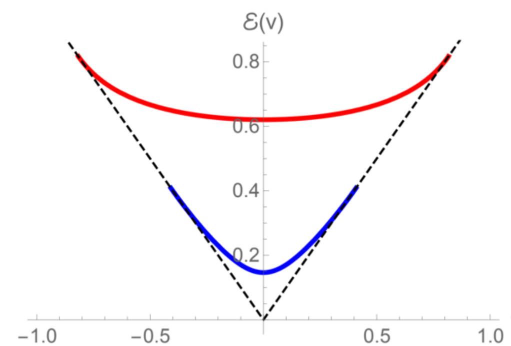
Mark Mezei , Membrane theory of entanglement dynamics from holography (2018)

MEMBRANE THEORY: TENSOR NETWORKS

Was first understood in 1+1D quantum mechanics







Tension for neutral (red) and charged (blue) black brane

Understood as `entanglement velocity' + minimisation membrane tension problem

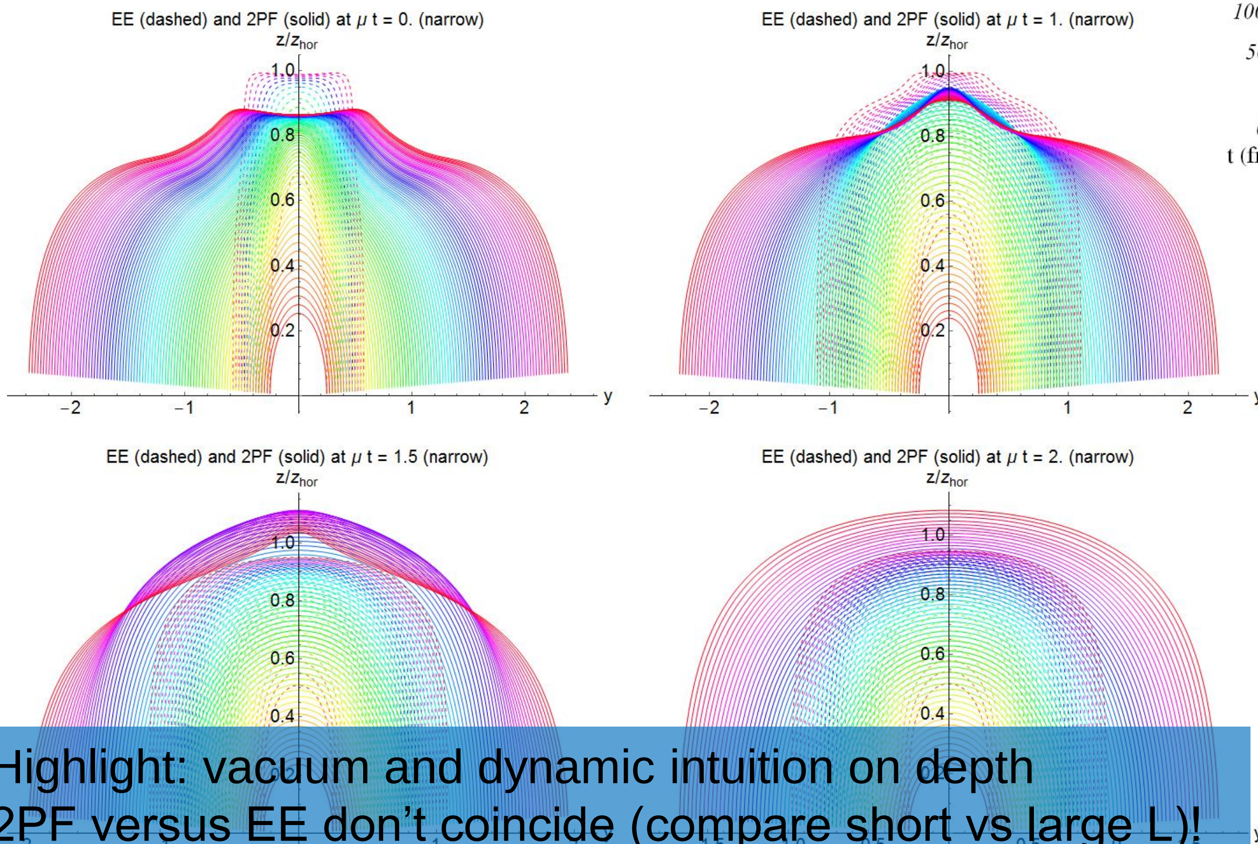
- Monotonic function (assuming ANEC in bulk)
- Interesting: $\mathcal{E}(0) = v_E\,, \quad \mathcal{E}'(0) = 0\,, \quad \mathcal{E}(v_B) = v_B\,, \quad \mathcal{E}'(v_B) = 1$
- (butterfly velocity can be related to horizon and OTOC)

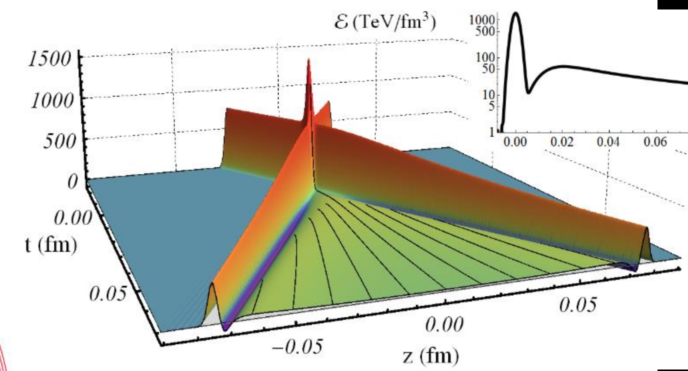
Cheryne Jonay, David A. Huse and Adam Nahum, Coarse-grained dynamics of operator and state entanglement (2018)
Márk Mezei and Douglas Stanford, On entanglement spreading in chaotic systems (2016)

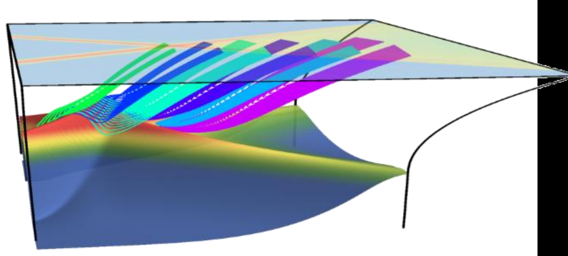
OUTLOOK

Non-homogeneous thermalisation: membrane theory as hydro

- For instance shock wave geometry:







Highlight: vacuum and dynamic intuition on depth
2PF versus EE don't coincide (compare short vs large L)!