Entanglement entropies of equilibrated pure states and the origin of replica wormholes

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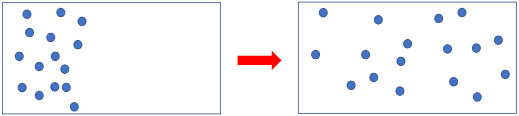
University of Ljubljana Holography Group Seminar, March 25th 2021
arXiv: 2008.01089
1912.08918, 2002.05734

Background and motivations: thermalization and unitarity

Equilibration

- Non-integrable many body systems are universally expected to approach equilibrium from an out-of-equilibrium initial state.



- Consider a quantum many-body system initially in a pure state $|\psi_0\rangle$.
$$|\psi(t)\rangle = U(t)\,|\psi_0\rangle$$

- At late times, $|\psi(t)\rangle$ should macroscopically resemble some equilibrium density matrix $\rho^{(\mathrm{eq})}$.

  For generic few-body observables O,
$$\langle\psi(t)|O|\psi(t)\rangle \approx \mathrm{Tr}[\rho^{(\mathrm{eq})} O]$$

Constraints from unitarity

- Under unitary evolution, pure states evolve to pure states. In particular, $\rho(t) = |\psi(t)\rangle \langle\psi(t)|$ cannot become equal to $\rho^{(\mathrm{eq})}$.

- Renyi entropies are one set of observables that can distinguish between pure and mixed states:
  (A is any subsystem, and $\bar{A}$ is its complement)

$$S_n^{(A)}(t) = -\frac{1}{n-1} \log \mathrm{Tr} \rho_A^n(t), \quad \rho_A(t) = \mathrm{Tr}_{\bar{A}} \rho(t), \quad n = 1, 2, \ldots$$

  In a pure state, for any n, we should have for any subsystem A

$$S_n^{(A)} = S_n^{(\bar{A})}.$$

  This is not true for a mixed state like $\rho^{(\mathrm{eq})}$.

- What form do the Renyi entropies of $|\psi(t)\rangle$ take at late times when the state has macroscopically equilibrated?

Black holes as thermalized pure states

Isolated star in pure state $|\psi_0\rangle$

$$\downarrow$$

Black hole

$$\downarrow$$

Black hole radiates (and often evaporates)

Black holes show macroscopic behaviour resembling thermal states:

- They emit thermal radiation at some temperature.
- Have an entropy satisfying standard thermodynamic relations.
- Correlation functions of few-body observables take thermal values.

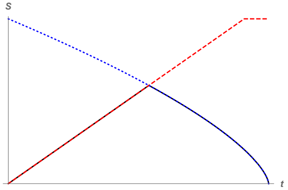
Black holes as thermalized pure states

- If black holes obey the unitarity of quantum mechanics, then the final state must be an equilibrated pure state.
- In particular, at all times, we should have

$$S_n^{(B)} = S_n^{(R)},$$

where B is the black hole and R is its emitted radiation.

- A calculation in semiclassical gravity by Hawking predicted that $S_n^{(R)} \gg S_n^{(B)}$ at late times. (S_R: red; S_B: blue)



- How can we obtain a unitary result for the entanglement entropies of an evaporating black hole?

Typical pure states

- Consider some system $L = A \otimes \bar{A}$. Average $S_n^{(A)}$ and $S_n^{(\bar{A})}$ over all pure states $|\psi\rangle \in L$ with the uniform measure. When either $d_A \ll d_{\bar{A}}$ or $d_{\bar{A}} \ll d_A$, Lubkin, Lloyd and Pagels, Page

$$\overline{S_n^{(A)}} = \overline{S_n^{(\bar{A})}} = \min\left(\log d_A, \log d_{\bar{A}}\right), \quad n = 1, 2, \cdots .$$

  If d_L is large, standard deviation about this average is small.

- Consistent with unitarity.

- When $d_A \ll d_{\bar{A}}$, the value of $S_n^{(A)}$ in typical pure states matches the value we would expect for a thermal state at infinite temperature, $\mathbf{1}/d_A$.

- How should this expression be generalized to finite temperature?

- Generalization to infinite-dimensional Hilbert spaces (e.g. QFTs)?

- Can we find a systematically improvable approximation for these quantities?

Plan

- Derivation of approximation method for finding entanglement entropies in equilibrated pure states.

  (Applicable for finite temperatures and infinite-dimensional Hilbert spaces; can in principle be systematically improved.)

- Mathematical structure and physical implications.
- Applications to the black hole information loss paradox.

Developing the approximation method

- Starting point: assume we can identify some equilibrium density matrix $\rho^{(\mathrm{eq})}$ that $|\psi(t)\rangle = U|\psi_0\rangle$ resembles at late times ($t \gg t_s$).

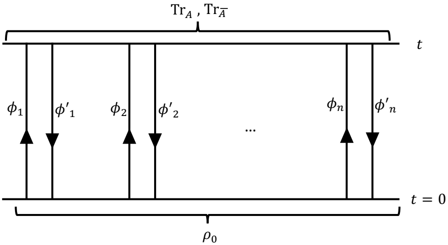
$$\rho^{(\mathrm{eq})} = \frac{1}{Z(\alpha)}\mathcal{I}_\alpha, \quad Z(\alpha) = \mathrm{Tr}\,\mathcal{I}_\alpha$$

- We will assume $Z(\alpha) \gg 1$.
- $\mathcal{I}_\alpha$ is invariant under the time-evolution, $U\mathcal{I}_\alpha U^\dagger = \mathcal{I}_\alpha$.
- Examples:
  - Infinite temperature:
  $$\mathcal{I} = \mathbf{1}, \quad Z = d$$
  - Microcanonical ensemble:
  $$\mathcal{I}_E = \sum_{E_n \in I} |n\rangle\langle n|, \quad Z(E) = \mathrm{Tr}\,\mathcal{I}_E = N_I, \quad I = (E - \Delta E, E + \Delta E)$$
  - Canonical ensemble:
  $$\mathcal{I}_\beta = e^{-\beta H}, \quad Z(\beta) = \mathrm{Tr} e^{-\beta H}$$

- We will now provide two kinds of expressions for the Renyi entropy as a transition amplitude on a replicated Hilbert space.
- Path integral representation:

$$\mathcal{Z}_n^{(A)} = e^{-(n-1)S_n^{(A)}} = \mathrm{Tr}_A \rho_A^n = \mathrm{Tr}_A \left(\mathrm{Tr}_{\bar{A}} U \rho_0 U^\dagger \right)^n$$

$$= (\dots) \prod_{i=1}^{n} \int_{\chi_i,\chi_i'}^{\psi_i,\psi_i'} D\phi_i(t) D\phi_i'(t) \exp\left(i \sum_{i=1}^{n} (S[\phi_i] - S[\phi_i']) \right)$$



- Exponent in path integral vanishes in configurations where Zhou and Nahum

$$\phi_i = \phi'_{\sigma(i)}, \quad \sigma \in S_n.$$

- Naively putting this into the path integral, we get a divergent answer.

Notation:

- σ, τ: permutations in S_n.
- e = identity permutation, $\eta = (n, n-1, ..., 1)$.
- For any operator $\mathcal{O}$ acting on $\mathcal{H}$, and any permutation $\sigma \in S_n$, define state $|\mathcal{O}, \sigma\rangle \in (\mathcal{H} \otimes \mathcal{H})^n$: $\quad$ ($|i\rangle$ is a basis for $\mathcal{H}$)

$$\langle i_1 \bar{i}'_1 i_2 \bar{i}'_2 \cdots i_n \bar{i}'_n | \mathcal{O}, \sigma\rangle = \mathcal{O}_{i_1 i'_{\sigma(1)}} \mathcal{O}_{i_2 i'_{\sigma(2)}} \cdots \mathcal{O}_{i_n i'_{\sigma(n)}}, \quad \mathcal{O}_{ij} = \langle i|\mathcal{O}|j\rangle$$

- Special case of $\mathcal{O} = \mathbf{1}$:

$$\langle i_1 \bar{i}'_1 i_2 \bar{i}'_2 \cdots i_n \bar{i}'_n | \sigma\rangle = \delta_{i_1 i'_{\sigma(1)}} \delta_{i_2 i'_{\sigma(2)}} \cdots \delta_{i_n i'_{\sigma(n)}} \ .$$

- $\langle e|\mathcal{O}, e\rangle = \mathrm{Tr}[\mathcal{O}]^n, \qquad \langle \eta|\mathcal{O}, e\rangle = \mathrm{Tr}[\mathcal{O}^n]$
- Similarly, can define $|O, \sigma\rangle_A \in (\mathcal{H}_A \otimes \mathcal{H}_A)^n$.

Using this notation, we can rewrite

$$\mathcal{Z}_n^{(A)} = \mathrm{Tr}_A \left(\mathrm{Tr}_{\bar{A}} U \rho_0 U^\dagger\right)^n = \langle \eta_A \otimes e_{\bar{A}}|(U \otimes U^\dagger)^n|\rho_0, e\rangle$$

- Define projector onto states spanned by $\{|\mathcal{I}_\alpha, \sigma\rangle\}_{\sigma \in S_n}$:
$$P_\alpha = \frac{1}{Z_2^n} \sum_{\sigma,\tau} g^{\sigma\tau} \, |\mathcal{I}_\alpha, \sigma\rangle \langle \mathcal{I}_\alpha, \tau| \ .$$
- Decomposing the identity on $(\mathcal{H} \otimes \mathcal{H})^n$ as
$$\mathbf{1} = P_\alpha + Q, \qquad P_\alpha Q = Q P_\alpha = 0, \qquad Q^2 = Q$$
we can rewrite
$$\mathcal{Z}_n^{(A)} = \langle \eta_A \otimes e_{\bar{A}} | (P_\alpha + Q)(U \otimes U^\dagger)^n (P_\alpha + Q) | \rho_0, e \rangle$$
- But from $U \mathcal{I}_\alpha U^\dagger = \mathcal{I}_\alpha$,
$$(U \otimes U^\dagger)^n P_\alpha = P_\alpha (U \otimes U^\dagger)^n = P_\alpha \ .$$
- Hence,
$$\mathcal{Z}_n^{(A)} = \langle \eta_A \otimes e_{\bar{A}} | P_\alpha | \rho_0, e \rangle + \langle \eta_A \otimes e_{\bar{A}} | Q (U \otimes U^\dagger)^n Q | \rho_0, e \rangle$$
$$\equiv \mathcal{Z}_{n,P}^{(A)} + \mathcal{Z}_{n,Q}^{(A)} \ .$$
Note that the first term is time-independent.
- Proposal: At late times, and in systems with large Hilbert space dimension, $\mathcal{Z}_{n,Q}^{(A)}$ can be ignored.

$$\mathcal{Z}_n^{(A)} \approx \mathcal{Z}_{n,P}^{(A)} = \frac{1}{Z_2^n} \sum_{\sigma,\tau} g^{\tau\sigma} \langle \eta_A \otimes e_{\bar{A}} | \mathcal{I}_\alpha, \tau \rangle \langle \mathcal{I}_\alpha, \sigma | \rho_0, e \rangle , \quad n = 1, 2, 3, \cdots \; .$$

Using the fact that the effective Hilbert space dimension Z_1 is large, we can simplify to:

$$\mathcal{Z}_n^{(A)} \approx \sum_{\tau \in \mathcal{S}_n} \mathcal{Z}_n^{(A)}(\tau), \qquad \mathcal{Z}_n^{(A)}(\tau) = \frac{1}{Z_1^n} \langle \eta_A \otimes e_{\bar{A}} | \mathcal{I}_\alpha, \tau \rangle \; .$$

In particular, note that the dependence on ρ_0 drops out, and the final expression is written only in terms of $\mathcal{I}_\alpha$.

Justification in infinite temperature case

- Exact expression:
$$\mathcal{Z}_n^{(A)} = \mathrm{Tr}_A \left(\mathrm{Tr}_{\bar{A}} U \rho_0 U^\dagger\right)^n = \langle \eta_A \otimes e_{\bar{A}} | (U \otimes U^\dagger)^n | \rho_0, e \rangle$$
- Averaging over all unitary time-evolution operators U with the Haar measure, we get the equilibrium approximation for infinite temperature Zhou and Nahum
$$\overline{(U \otimes U^\dagger)^n} = P_{\mathcal{I}=\mathbf{1}} = \frac{1}{d^n} \sum_{\sigma,\tau} g^{\sigma\tau} \left|\sigma\right\rangle \left\langle\tau\right|$$
- Consider the variance under the Haar average:
$$\overline{\left(\mathcal{Z}_{n,Q}^{(A)}\right)^2} = \overline{(\mathcal{Z}_n^{(A)})^2} - (\mathcal{Z}_{n,P}^{(A)})^2.$$
- In the large d limit,
$$\overline{\left(\mathcal{Z}_{n,Q}^{(A)}\right)^2} \ll (\mathcal{Z}_{n,P}^{(A)})^2.$$
- For other choices of $\rho^{(\mathrm{eq})}$, we do not have a similar average over relevant time-evolution operators.

Self-consistency criterion

- For any time-evolved quantity that can be written in the form

$$F(t) = \langle A| \; (U \otimes U^{\dagger})^{q} \; |\rho_0, e\rangle$$

we can define the equilibrium approximation

$$[F]_{\text{eq app}} = \langle A| \; P_{\alpha} \; |\rho_0, e\rangle$$

- To see if the approximation is valid for a quantity F, we use the self-consistency criterion

$$[F^2]_{\text{eq app}} - ([F]_{\text{eq app}})^2 \ll ([F]_{\text{eq app}})^2.$$

- For the Renyi entropies, we can show

$$\Delta \equiv \left[\left(\mathcal{Z}_n^{(A)}\right)^2\right]_{\text{eq app}} - \left(\mathcal{Z}_{n,P}^{(A)}\right)^2 \ll \left(\mathcal{Z}_{n,P}^{(A)}\right)^2$$

when the effective Hilbert space dimension is large.

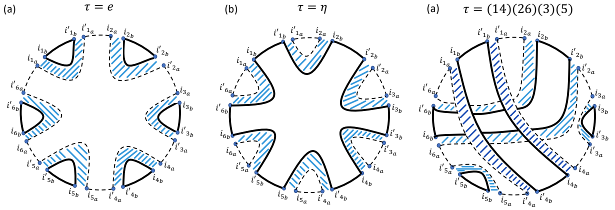
Equilibrium approximation for other observables

- Various other quantities can be expressed as transition amplitudes on replicated Hilbert spaces:
  n-point functions of observables, spectral form factors, Renyi negativities, and Renyi relative entropies.
- To see whether the equilibrium approximation is valid, need to check self-consistency criterion.
- **Expectation values of observables:**
  $$\langle O(t)\rangle_{\text{eq app}} = \text{Tr}[\rho_{\text{eq}} O]$$
  If O has support only on subsystem A, then the self-consistency criterion is met if $d_A \ll d_{\bar{A}}$, otherwise not valid.
- $n-$**th power of spectral form factor:**
  $$\mathcal{S}_n \equiv (\text{Tr}U\text{Tr}U^\dagger)^n = \hat{\text{Tr}}[(U \otimes U^\dagger)^n]$$
  For any choice of $\mathcal{I}_\alpha$, we get
  $$[\mathcal{S}_n]_{\text{eq app}} = n!$$
  So the self-consistency criterion is not met for any n. (Also, does not capture expected late-time features of ramp, plateau.)

Mathematical structure and physical consequences

Diagrammatic interpretation

$$\mathcal{Z}_n^{(A)} \approx \sum_{\tau \in \mathcal{S}_n} \mathcal{Z}_n^{(A)}(\tau), \qquad \mathcal{Z}_n^{(A)}(\tau) = \frac{1}{Z_1^n} \langle \eta_A \otimes e_{\bar{A}} | \mathcal{I}_\alpha, \tau \rangle \ .$$



$k(\tau)$: number of cycles in τ, number of solid loops in diagram.
$k(\eta\tau^{-1})$: number of dashed loops.

$$\mathcal{Z}_n^{(A)}(\tau) \sim \frac{1}{Z_1^n} d_A^{k(\eta^{-1}\tau)} d_{\bar{A}}^{k(\tau)}$$

For any τ,

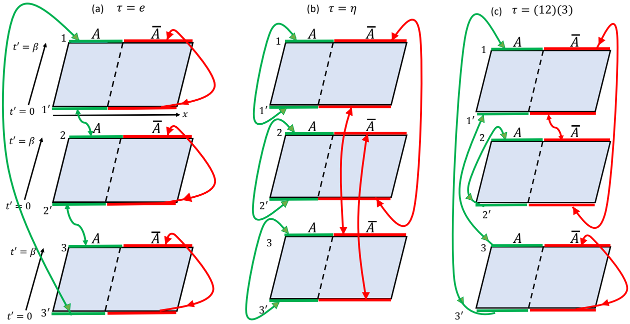
$$k(\eta^{-1}\tau) + k(\tau) \leq n + 1 \ .$$

Saturated for non-crossing permutations/ planar diagrams.

Path integral representation

$$\mathcal{Z}_n^{(A)} \approx \frac{1}{Z_1^n} \sum_{\tau \in \mathcal{S}_n} \langle i_{\eta(1)_a} i_{1_b} | \mathcal{I}_\alpha | i_{\tau(1)_a} i_{\tau(1)_b} \rangle \cdots \langle i_{\eta(n)_a} i_{n_b} | \mathcal{I}_\alpha | i_{\tau(n)_a} i_{\tau(n)_b} \rangle$$

$$= \frac{1}{Z_1^n} \sum_{\tau \in \mathcal{S}_n} \int \prod_{i=1}^{n} D\psi_i D\psi_i' \, \delta(\psi_{iA} - \psi'_{\tau^{-1}\eta(i)A}) \delta(\psi_{i\bar{A}} - \psi'_{\tau^{-1}(i)\bar{A}}) \prod_{i=1}^{n} \int_{\psi_i'}^{\psi_i} D\phi_i \, e^{-S_E[\phi_i]}$$



A sum of $|S_n|$ Euclidean path path integrals on n copies of the system emerges as an approximation to the original Lorentzian path integral on $2n$ copies.

Universal implications

- Unitarity constraint is satisfied:

$$\mathcal{Z}_n^{(A)} = \sum_\tau \mathcal{Z}_n^{(A)}(\tau) = \frac{1}{2}\sum_\tau \left(\mathcal{Z}_n^{(A)}(\tau) + \mathcal{Z}_n^{(\bar{A})}(\tau)\right)$$

  so that $\mathcal{Z}_n^{(A)} = \mathcal{Z}_n^{(\bar{A})}$.

- Dominant contributions in various regimes:

$$\mathcal{Z}_n^{(A)}(\tau) \sim \frac{1}{Z_1^n}\, d_A^{k(\eta^{-1}\tau)} d_{\bar{A}}^{k(\tau)}$$

  - (i) When $d_A \ll d_{\bar{A}}$, the $\tau = e$ term dominates,

$$\mathcal{Z}_n^{(A)} \approx \mathrm{Tr}_A\left[\left(\mathrm{Tr}_{\bar{A}}\rho_{\mathrm{eq}}\right)^n\right]$$

    (ii) When $d_{\bar{A}} \ll d_A$, the $\tau = \eta$ term dominates,

$$\mathcal{Z}_n^{(A)} \approx \mathrm{Tr}_{\bar{A}}\left[\left(\mathrm{Tr}_A\rho_{\mathrm{eq}}\right)^n\right]$$

    Expected generalization of Page's result:

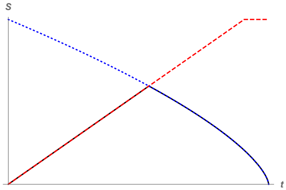
$$S_n^{(A)} = \min\left(S_n^{(A,\mathrm{eq})}, S_n^{(\bar{A},\mathrm{eq})}\right), \quad n \geq 2 \quad \Rightarrow S_1^{(A)} = \min\left(S_1^{(A,\mathrm{eq})}, S_1^{(\bar{A},\mathrm{eq})}\right)$$

  - When $d_{\bar{A}}$ and d_A are both large and comparable, the non-crossing permutations/planar diagrams dominate.

Application to the black hole information loss paradox

The black hole information problem

- Key to showing that black holes do not destroy information despite looking thermal: show that $S_n^{(B)} = S_n^{(R)}$ at all times.
- A calculation in semiclassical gravity by Hawking predicted that $S_n^{(R)} \gg S_n^{(B)}$ at late times. (S_R: red; S_B: blue)

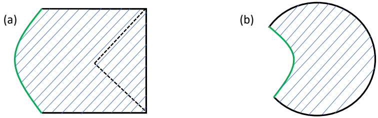


- Previously, it was expected that Hawking's information loss paradox could only be resolved using non-perturbative quantum gravity.
- Recent works Penington; Almheiri, Engelhardt, Marolf, Maxfield; Almheiri, Mahajan, Maldacena, Zhao showed that a semiclassical prescription with some new ingredients is sufficient to get a unitary result.

A simple model for black hole evaporation

Penington, Shenker, Stanford, Yang

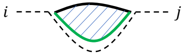
- After the evaporation process, we have a state $|\Psi\rangle \in B \otimes R$.
  $B =$ black hole, $R =$ radiation.
- Black hole lives in a (1+1)-D spacetime with JT gravity, with end-of-the-world (EOW) branes behind the horizon.



- R has N orthonormal states $|i\rangle$.
- Evaluate $(\rho_R)_{ij} = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)_{ij}$ using a Euclidean gravitational path integral:
  - Boundary condition is a segment of length β.

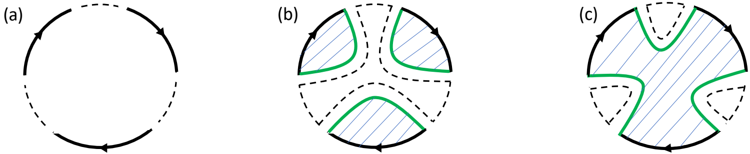
    i and j give b.c. for the state on the EOW brane.
  - This boundary condition is "filled in" with a bulk geometry. (Joining dotted lines gives δ_{ij}.)

A simple model for black hole evaporation

Penington, Shenker, Stanford, Yang

- Now to evaluate, for instance, $\mathcal{Z}_3^{(R)} = \mathrm{Tr}[\rho_R^3] = (\rho_R)_{ij}(\rho_R)_{jk}(\rho_R)_{ki}$



- Geometries like (c), where the Euclidean path integral connects different replicas, are important for ensuring $S_B = S_R$. "Replica wormholes."
- Why should the entanglement entropies of the state formed from the black hole collapse and evaporation be evaluated in this way?
- How do such Euclidean path integrals emerge from the Lorentzian dynamics?
- Why does including such geometries ensure unitarity?

Equilibrium approximation for the evaporation model

- Initial pure state $|\Psi_0\rangle$.
- Final state $|\Psi\rangle = U(t)\,|\Psi_0\rangle$ macroscopically resembles $\rho^{(\text{eq})}$ with

$$\mathcal{I}_\alpha = \mathbf{1}_R \otimes \mathcal{I}^{(B)}_\beta, \quad \mathcal{I}^{(B)}_\beta = e^{-\beta H_B}$$

  (Radiation has no energy constraint, dimension N; black hole associated with temperature β.)

- Applying the equilibrium approximation, we get

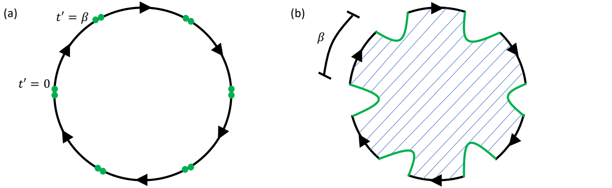
$$\mathcal{Z}^{(R)}_n \approx \frac{1}{\left(N Z^{(B)}_1\right)^n} \sum_\tau N^{k(\tau\eta^{-1})} Z^{(B)}_{n_1} \cdots Z^{(B)}_{n_{k(\tau)}}$$

  where $k(\tau)$: number of cycles in τ;
  $n_1, \dots, n_{k(\tau)}$: lengths of cycles of τ, and

$$Z^{(B)}_m = \mathrm{Tr}[e^{-m\beta H_B}] = e^{S_0} z_m(\beta)$$

- $Z^{(B)}_n$ for each n corresponds to a boundary Euclidean path integral on n copies. Using holography, equal to a bulk path integral.
- e^{S_0} and N both large $\to$ non-crossing permutations dominate.

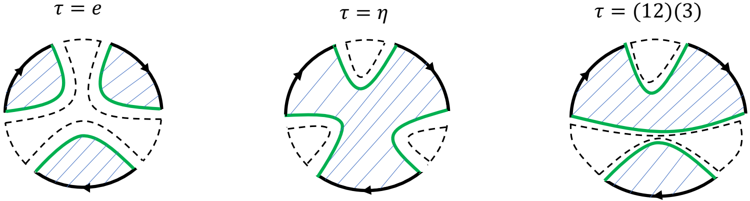
- To match the EOW brane model, $\mathcal{I}_\alpha = \mathbf{1}_R \otimes f(H_B)\mathcal{I}_\beta^{(B)}$.
- Then the bulk and boundary path integrals for calculating $Z_m^{(B)}$



- One-to-one correspondence between contributions from non-crossing permutations τ in

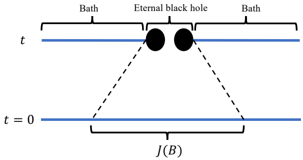
$$\mathcal{Z}_n^{(R)} \approx \frac{1}{\left(NZ_1^{(B)}\right)^n} \sum_\tau N^{k(\tau\eta^{-1})} Z_{n_1}^{(B)} \cdots Z_{n_{k(\tau)}}^{(B)}$$

and planar diagrams contributing in the same limit according to the prescription of Penington, Shenker, Stanford, Yang :



General conclusions

- The equilibrium approximation provides a derivation for the replica wormhole prescription for calculating $\mathcal{Z}_n^{(R)}$.
- Unconventional geometries for $\mathcal{Z}_n^{(R)}$ come from $\tau \neq e$ terms in the Euclidean path integral expression for the equilibrium approximation.
- Unitary result on very general grounds from our earlier argument.
- Can be applied to other black hole models, including:
  - Eternal black hole coupled to an infinite bath. Almheiri, Hartman, Maldacena, Shaghoulian, Tajdini



  - Non-evaporating black holes, such as big black holes in AdS.

Precise prescriptions for when and how replica wormholes should be included.

Non-factorization of matrix elements

- Penington, Shenker, Stanford, Yang noted that including replica wormholes leads to results that suggest that $\langle|(\rho_R)_{ij}|^2\rangle$ does not factorize.
- We find the same issue using the equilibrium approximation: Approximation for matrix elements:
$$(\rho_R)_{ij} = (\rho_R^P)_{ij} + \Delta_{ij}^Q, \qquad (\rho_R)_{ij} \approx (\rho_R^P)_{ij} = \frac{1}{N}\delta_{ij}$$
Approximation to the second Renyi entropy:
$$\mathcal{Z}_2^{(R)} = \sum_{ij} |(\rho_R)_{ij}|^2 \approx \frac{1}{N} + \frac{1}{e^{S_0}} \times O(1)$$
So the two expressions are incompatible when $N \gg e^{S_0}$.
- Using the self-consistency condition for $(\rho_R)_{ij}$, we can estimate
$$|\Delta_{ij}^Q|_{\text{eq app}} \sim \frac{1}{N} e^{-S_0/2}$$
    - Individual off-diagonal matrix elements, coming from fluctuations around the equilibrium value of $(\rho_R)_{ij}$, are suppressed.
    - They make a significant contribution to $\mathcal{Z}_2^{(R)}$ due to their large number.

Semiclassical gravity and averaging

- It has been suggested that the lack of factorization implies that semiclassical gravity computes an average over theories,

$$\overline{|\rho_{ij}|^2} \neq \overline{(\rho_{ij})}\ \overline{(\rho_{ij})^*}$$

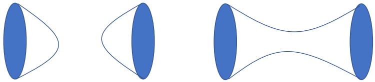
- The equilibrium approximation gives an explanation based on fluctuations around equilibrium. Valid for a theory with a single fixed Hamiltonian, no need for average over theories.

- Our derivation of the replica wormholes consisted of two steps:
  (1) Approximating Lorentzian expressions with a sum of Euclidean path integrals.
  (2) Using semiclassical gravity to compute the Euclidean path integrals.
  The problem with factorization arose at the first step.

- So in principle, the issue can be resolved within semiclassical gravity, by doing a Lorentzian calculation.

Comparison to wormholes in spectral form factors

- In Saad, Shenker, Stanford, trumpet contributions were found for the spectral form factor $\mathrm{Tr}[e^{iHt}]\mathrm{Tr}[e^{-iHt}]$ in Euclidean JT gravity,

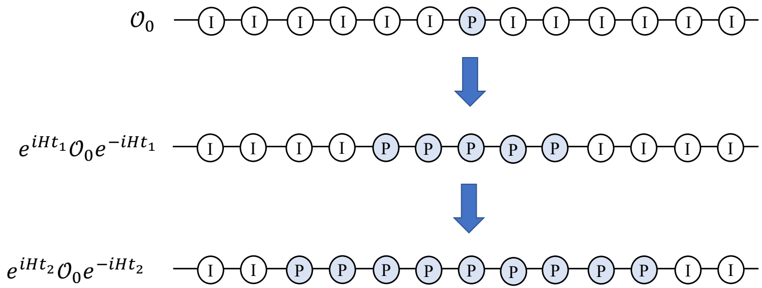


  which lead to the non-factorization of this quantity into the product of $\mathrm{Tr}[e^{iHt}]$ and $\mathrm{Tr}[e^{iHt}]$ computed by the same method.
- As discussed earlier, this quantity is beyond the scope of the equilibrium approximation.
- Note that here the Lorentzian path integral on 2 copies of the system is being evaluated with the help of a Euclidean path integral on 2 copies: this does not have the same structure as the equilibrium approximation.
- The "wormholes" appearing in these two scenarios are thus physically distinct.

Thermalization from an operator evolution perspective

Operator growth in chaotic systems

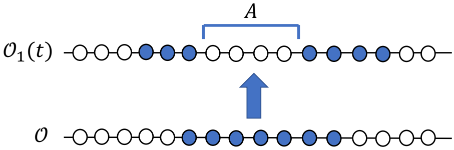
- Consider an operator which initially has support in a small subsystem of a chaotic system.
- The support of this operator increases with time (leads to decay of OTOCs). Susskind, Shenker, Roberts, Stanford



Void formation

Liu and SV, 1912.08918

- Within the growing support of the operator, it has some probability of being equal to the identity in any given region.



- At any time t, with respect to any region A, write $O(t)$ as

$$\mathcal{O}(t) = \mathcal{O}_1(t) + \mathcal{O}_2(t), \qquad \mathcal{O}_1(t) = \tilde{O}_{\bar{A}} \otimes \mathbf{1}_A, \qquad \mathrm{Tr}_A[\mathcal{O}_2(t)] = 0$$

- We refer to the presence of $\mathcal{O}_1(t)$ in $\mathcal{O}(t)$ as void formation from $\mathcal{O}$ in A.

Should expect this probability to be small at late times in chaotic systems, since the support of operators tends to grow.

However, we find that in any many-body system, void formation processes have important consequences:

- Ensuring unitarity of entanglement growth.
- Generation of multipartite entanglement between disjoint regions.
- Explaining differences in entanglement growth between integrable and chaotic systems.

Equilibration and the random void distribution

- Write the initial density matrix as

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{d}\mathbf{1} + \hat{\rho}, \quad \mathrm{Tr}[\hat{\rho}] = 0$$

- If we assumed that the probability of void formation of $\hat{\rho}$ in $\bar{A}$ is negligible at $t \gg t_s$, then we would find

$$S_n^{(A)} = \log d_A, \quad S_n^{(A)} \neq S_n^{(\bar{A})}$$

- Recall that from the equilibrium approximation at infinite temperature, if $d_A \ll d_{\bar{A}}$ or $d_{\bar{A}} \ll d_A$

$$S_n^{(A)} = S_n^{(\bar{A})} = \min(\log d_A, \log d_{\bar{A}})$$

- Thus, to include the contributions from $\tau \neq e$ that lead to a unitary result, from an operator growth perspective we need to take void formation processes into account.

Random void distribution

- In particular, if $\hat{\rho}_1(t)$ is the part of $\hat{\rho}(t)$ with a void in $\bar{A}$, to get

$$S_n^{(A)} = S_n^{(\bar{A})} = \min(\log d_A, \log d_{\bar{A}})$$

we need

$$P_{\hat{\rho},n}^{(\bar{A})} \equiv \frac{\mathrm{Tr}[(\hat{\rho}_1(t))^n]}{\mathrm{Tr}[\hat{\rho}^n]} = \frac{1}{d_{\bar{A}}^{2(n-1)}}, \quad \bar{A} \ll A, \quad n \geq 2$$

- From a calculation in chaotic local random unitary circuits, we find that $P_{O,n}^{(\bar{A})}$ takes this form for any traceless operator O.

- Thermalization to infinite temperature in unitary systems can be seen as a consequence of this general property of operator growth, which we call the “random void distribution.”

Further directions

- Applying the method to other kinds of observables, such as Renyi negativity and Renyi relative entropy. work in progress with J. Kudler-Flam and H. Liu

  Helps understand effects of finite temperature on transfer of information from the black hole to the radiation, and correlations within the radiation.

- Systematically understanding corrections to the approximation from the $\mathcal{Z}_Q$ term, and the time-scale for approaching equilibrium. work in progress with Z.D. Shi and H. Liu

- Can equilibration of pure states can be related to a universal property of operator growth in the finite temperature case?

Thank you!

Example: Infinite temperature

- Taking $\mathcal{I}_\alpha = \mathbf{1}$, and considering the limit where d_A and $d_{\bar{A}}$ are large and comparable,

$$\mathcal{Z}_n^{(A)} = \frac{1}{d^n}\sum_{k=1}^{n} N(n,k) d_A^{n+1-k} d_{\bar{A}}^{k}$$

$$= \frac{1}{d_{\bar{A}}^{n-1}} + \frac{1}{2}\frac{n(n-1)}{d_A d_{\bar{A}}^{n-2}} + \cdots + \frac{1}{2}\frac{n(n-1)}{d_A^{n-2} d_{\bar{A}}} + \frac{1}{d_A^{n-1}}$$

  $N(n,k)$: number of non-crossing partitions of n objects with k blocks/ Narayana numbers.

- By analytic continuation, we can recover Page's result

$$S_1^{(A)} = \begin{cases} \log d_A - \frac{1}{2}\frac{d_A}{d_{\bar{A}}} & d_A < d_{\bar{A}} \\ \log d_{\bar{A}} - \frac{1}{2}\frac{d_{\bar{A}}}{d_A} & d_{\bar{A}} < d_A \end{cases} .$$

- From equilibrium approximation: we can in principle systematically improve this result by including contributions from $\mathcal{Z}_Q$.

Canonical and microcanonical ensemble

- Taking $\mathcal{I}_\alpha = \mathcal{I}_E, \ \mathcal{I}_\beta$, we get universal results for pure states that equilibrate to these ensembles, applicable to any quantum many-body system.
- These results agree with averages over "ergodic bipartition" states Lu and Grover and "canonical thermal pure states" Nakagawa, Watanabe, Sugiura, Fujita.
- Unlike Page's calculation, these averages cannot be seen as averages over time-evolution operators.
- We find the same results as approximations to the time-evolved expressions, and also see how they come from a common structure.
- So far, we have assumed there is some time-scale t_s after which the full system thermalizes. Not true for uncompact systems with local interactions.

  From a simple modification for systems with a sharp light-cone:

$$S_n^{(A)}(t) = s_n^{\mathrm{eq}} \min(|A|, 2t)$$

Approximation for Renyi and von Neumann entropies

- We can further show that

$$\left[\left(\mathcal{Z}_n^{(A)}\right)^m\right]_{\text{eq app}} - \left(\mathcal{Z}_{n,P}^{(A)}\right)^m \ll \left(\mathcal{Z}_{n,P}^{(A)}\right)^m , \quad m = 2, 3, \ldots$$

- Through analytic continuation, we get the equilibrium approximation for the Renyi entropies,

$$\begin{aligned}
[S_n^{(A)}]_{\text{eq app}} &= -\frac{1}{n-1}\lim_{m\to 0}\frac{\partial[(\mathcal{Z}_n^{(A)})^m]_{\text{eq app}}}{\partial m} \\
&\approx -\frac{1}{n-1}\lim_{m\to 0}\frac{\partial(\mathcal{Z}_{n,P}^{(A)})^m}{\partial m} = -\frac{1}{n-1}\log(\mathcal{Z}_{n,P}^{(A)}) \ . \qquad (1)
\end{aligned}$$

- Similarly, for the von Neumann entropy:

$$[S_1^{(A)}]_{\text{eq app}} = -\lim_{n\to 1}\frac{\partial \mathcal{Z}_{n,P}^{(A)}}{\partial n} \ .$$